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SOME NOTES ON THE DESCRIPTION OF JET NOISE SOURCE TERMS

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I. INTRODUCTION

Lilley's equation, exploring the analogy with the linear propagation of sound in a unidirectional transversely sheared flow [1], has been successfully employed for the description of jet noise. Although there is no question regarding the form of the linear operator, the representation of the nonlinear source terms is not unique. It is remarkable that Tester and Morfey [2], who produced one of the most complete analysis on the equation and its solutions, departed from what could be called the 'standard' source function which, for isothermal jets, is usually interpreted as a volume acceleration quadrupole distribution, introducing, instead, volume displacement sources. Because of this, their results cannot be directly compared with those of other authors. The present paper discusses the arguments that led to this choice, as well as characteristics of the solutions and their relevance to the jet noise problem, extending the analysis presented in [3].

2. SOURCES IN SHEAR FLOWS

The equation governing sound generation and propagation in a unidirectional sheared mean flow, with x_1 as the flow direction, if there are sources of mass and momentum per unit mass, q and f, the mean values of velocity U and sound speed c_0 are both functions of x_2 and x_3 only and viscosity and heat conduction are neglected, can be writen

$$(\rho_0 c_0^2)^{-1} \mathscr{L} p = Q \tag{1}$$

with

$$\mathscr{L} = \frac{D_0}{Dt}\{\frac{D_0^2}{Dt^2} - \nabla.(c_0^2\nabla)\} + 2c_0^2\nabla U.\nabla\frac{\partial}{\partial x_1} \tag{2}$$

$$Q = \{-\frac{D_0}{Dt}\nabla. + 2\nabla U.\frac{\partial}{\partial x_1}\}f + \frac{D_0^2}{Dt^2}q \tag{3}$$

where p is the pressure fluctuation, ρ_0 is the mean density and $D_0/Dt = \partial/\partial t + U\partial/\partial x_1$.

The dipole distribution f is actually of quadrupole character when $f = -\nabla.T$. These source distributions, stemming from the *momentum* equation are of the *volume acceleration* type. The monopole distribution q, which originates from the *mass balance* equation, is a volume velocity source. A *volume displacement* monopole distribution, q^*, is obtained for $q = D_0q^*/Dt$. Higher order volume displacement multipoles, f^*, T^*, ..., can be generated for $q^* = -\nabla.f^*$, $q^* = \nabla.\nabla.T^*$, and so on. For these source distributions the source function does not present a term dependent on mean shear, as it does for the volume acceleration ones.

The distinction between both types of sources will be important when there is movement, be it of the surrounding flow and/or of the source itself, as the respective solutions may present significant differences for the same multipole order. The sources that display the intuitive idea of movement are the volume displacement ones, since they can be related to the fluid elements, under force or stress, for instance, that move. For moving volume acceleration sources, it is the action (the force or the stress field) that moves, the fluid remaining at rest, as in Lighthill's original analogy, unless the medium also moves, as in the situation described by Lilley's equation.

3. SOURCE TERMS OF LILLEY'S EQUATION

It is quite common to find in the literature Lilley's equation writen as

$$\mathscr{L}p = \ldots$$

where the ... stand for the nonlinear source terms, what may suggest, at first sight, that they form a long and complicated function of the fluctuating quantities. This view even appears to be justified if one looks as the full nonlinear equation before any 'linearization' is attempted, given by

$$\left[\frac{D}{Dt}\left\{\frac{D^2}{Dt^2} - \frac{\partial}{\partial x_i}\left(c^2 \frac{\partial}{\partial x_i}\right)\right\} + 2\,\frac{\partial v_i}{\partial x_j}\frac{\partial}{\partial x_i}\left(c^2 \frac{\partial}{\partial x_j}\right)\right]\pi = -2\,\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_i} \tag{4}$$

where $\pi = \gamma^{-1}\ln(P/P_\infty)$, γ is the specific heat ratio, P,**v**, and c refer to the total values of pressure, velocity and sound speed, respectively, P_∞ is a reference value for P and $D/Dt = \partial/\partial t + v_i\partial/\partial x_i$.

If, on the other hand, the fundamental equations are writen from the outset as non–homogeneous *linear* diferential equations, the nonlinear portions being interpreted as source terms, it is quite straightforward to obtain

$$\mathscr{L}\pi = Q \tag{5}$$

where Q is given by (3) with

$$\mathbf{f} = -(\mathbf{u}.\nabla\mathbf{u}) - c^{2\prime}\nabla\pi \tag{6}$$

$$q = -\mathbf{u}.\nabla\pi \tag{7}$$

where **u** and $c^{2\prime}$ represent the fluctuating parts of **v** and c^2. Note that π fluctuations can be approximated by $p/(\rho_0 c_0^2)$, the product $\rho_0 c_0^2$ being considered as a constant.

It is usual to neglect the monopole term (7) and to write the first term in the dipole distribution (6) as $-\nabla.(\mathbf{u}\otimes\mathbf{u})$. The last procedure can be justified if $\mathbf{u}$ is taken as the solenoidal part of the velocity fluctuation vector, while the (doubtful) assumption $p \sim O(u_i^2)$ can answer for both approximations. The true dipole term, related to fluctuations in c^2, was identified by Tester and Morfey [3] and corresponds to a similar term in the expansion of the sources for Lightill's equation, identified by Morfey [4].

It was perharps Goldstein [5] the first to identify the main source term for isothermal jets with a (volume acceleration) quadrupole distribution per unit mass in a shear flow. He was also able to establish, by ingenious manipulation of the source terms [6], that the second order modified (adimensional) pressure fluctuation $\Pi' = \pi_{(2)} + (1/2)\,\pi_{(1)}{}^2$ – here and elsewhere the subscripts $_{(1)}$ and $_{(2)}$ refer to first and second order fluctuations, respectively – which provides a better approximation for the isentropic density fluctuations than either $p_{(2)}/(\rho_0 c_0^2)$ or $\pi_{(2)}$, satisfies

$$\mathscr{L}\,\Pi' = \{-\frac{D_0}{Dt}\nabla. + 2\nabla U.\frac{\partial}{\partial x_1}\}\mathbf{f}_{(2)} \tag{8}$$

$$\mathbf{f}_{(2)} = -\nabla.(\mathbf{u}_{(1)}\otimes\mathbf{u}_{(1)}) - c^2{}_{(1)}\nabla\pi_{(1)} \tag{9}$$

The second order source function involves only products of first order quantities, while $\pi_{(1)}$ satisfies $\mathscr{L}\pi_{(1)} = 0$.

Thus Goldstein fully justified what had become the standard procedure of modelling the source terms for isothermal jets as the volume acceleration quadrupole distribution, $T_{ij} = u_i u_j$. A significant exception for this procedure (made, indeed, before it could be called standard) was due to Tester and Morfey [2], who wrote Lilley's equation in a form equivalent to

$$\mathscr{L}\pi = \frac{D_0}{Dt}\{\frac{3}{2}\nabla.\nabla.(\mathbf{u}\otimes\mathbf{u}) + \nabla.(c^{2\prime}\nabla\pi)\} - 2\nabla U.\frac{\partial}{\partial x_1}(c^{2\prime}\nabla\pi) + O(u_i^3) \tag{10}$$

As the quadrupole–like term $3/2D_0(\nabla.\nabla.(\mathbf{u}\otimes\mathbf{u}))/Dt$ did not conform with the expected U^8 law for acoustic intensity, they choosed, in a somewhat arbitrary way, to describe the sources as of the volume displacement type, rewriting the equation as

$$(\rho_0 c_0^2)^{-1}\mathscr{L}p = \frac{D_0^3}{Dt^3}(\nabla.\nabla.\mathbf{T}^* - \nabla.\mathbf{f}^*) \tag{11}$$

where $\mathbf{T}^*$ and $\mathbf{f}^*$ represent volume displacement quadrupole and dipole distributions, the relationship between the corresponding terms in (10) and (11) being given, to second order in u_i, in [2]. As the source terms in (5–9), (10) and (11) correspond to different physical interpretations, it is necessary to review the steps that lead from one to the other.

4 DISCUSSION

From the analysis of section 2, it is evident that the quadrupole–like term in (10), although containing a double divergence, cannot be identified with a volume acceleration quadrupole distribution, due to the absence of the ∇U term. Indeed, as it does not posses the physical meaning of any of the discussed types of quadrupole distributions, it is not surprising that it does not conform to expected quadrupole properties. This fact led to the option for volume displacement sources, which yielded again a source with a sound physical meaning. Even so, as jet noise sources, both the quadrupole and the dipole ones, originate mainly from the *momentum* equation, their description as volume displacement sources is actually inappropriate.

The origin of the numerical factor 3/2 in the pseudo volume acceleration quadrupole in equation (10) can be understood from the fact that, for the mean flow under discussion, the right hand side of (4) can be writen as

$$-2\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_i} = Q = -2\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_i} + 3\frac{\partial U}{\partial x_j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_1}\right) \tag{12}$$

and also as

$$Q = \frac{D}{Dt}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) - 2\frac{\partial U}{\partial x_j}\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_1} + 2\frac{\partial u_j}{\partial x_i}\frac{\partial}{\partial x_j}\left(c^2\frac{\partial \pi}{\partial x_i}\right) + 2\frac{\partial^2 U}{\partial x_i \partial x_j}\frac{\partial u_j}{\partial x_1}u_i \tag{13}$$

Thus, to obtain the source function in (10) it is necessary, besides neglecting the third order terms and considering u as the rotational part of the velocity fluctuation vector, to neglect also the difference between one and a half times the last two terms in (13) and the nonlinear terms (other than those associated with $c^{2\prime}$) in the left hand side of (4), that must also be treated as source terms. In doing this, second order terms that look small and do not have a physical meaning of their own as a source, but which contribute to building up source terms that do have such a meaning, are abandoned, leading up to sources that are non–physical, as the quadrupole–like term in (10). Further use of 'small' (third order) terms permitted the obtention of volume displacement sources which, for the jet noise problem, are also non–physical. This draws attention to the objection sometimes made to Lilley's equations, that it treats turbulence as a small perturbation. That this is not true can be recognized from equations (5–7) and even more clearly if, following the suggestion inherent in Goldstein's formulation (8–9), one looks for the corresponding equation when no approximation is made. Thus, for $\Pi = \pi + (1/2)\pi^2$ one obtains

$$\mathscr{L}\Pi = \left\{-\frac{D_0}{Dt}\nabla. + 2\nabla U.\frac{\partial}{\partial x_1}\right\}f + \frac{D_0^2}{Dt^2}q \tag{14}$$

$$f = -\nabla.(\mathbf{u}\otimes\mathbf{u}) - c^{2\prime}\nabla\Pi - \mathbf{u}.\nabla(\mathbf{u}\pi), \quad q = -\mathbf{u}.\nabla\pi^2/2 \tag{15–a,b}$$

Equations (5–7) and (14–15) show that it is a property of Lilley's equation that all nonlinear source terms combine in such a way as to have a clear interpretation as sources in shear flows, *whith no term of $O(u_i^3)$ involving only velocity fluctuations.* Both the monopole terms in (7) and (15) or the almost quadrupole term in (15), coupling **u** and π, are mainly related to turbulent scattering of sound and to other second (or higher) order *propagation* effects, what justifies their neglect. Of course the dipole $-\mathbf{u}.\nabla(\mathbf{u}\pi)$ and the monopole $-\mathbf{u}.\nabla\pi^2/2$ are more comfortably discarded than the monopole (7), which does have a second order part.

Thus, it seems that more important than an order analysis of the source terms of Lilley's equation is writing the terms in such a way as to emphasize their physical meaning in the generation and propagation processes and as sources in a shear flow. If this is done, as in [6], the order analysis will yield extremely useful additional information.

To complete the analysis, the characteristics of the solutions due to both types of sources are discussed bellow.

5. REPRESENTATION OF THE SOLUTIONS

To gain insight into the solutions of Lilley's equations (and indeed into the equation itself), it is most useful to work with a modified form of the operator, introduced in [3]. As long as far field solutions are concerned, it is acceptable to substitute the $\partial/\partial x_1$ derivatives *on both sides* of the equation by its far field form, $-(1/c_\infty)(x_1/|\mathbf{x}|)\ \partial/\partial t = -(\cos\theta/c_\infty)\ \partial/\partial t$, where c_∞ is the far field value of c_0 and $\cos\theta = x_1/|\mathbf{x}|$ is to be treated as a constant. This permits rewriting equation (1) as

$$\frac{\partial}{\partial t}\left[C\{C^2\frac{\partial^2}{\partial t^2} - \nabla.(c_0^2\nabla)\} - 2c_0^2\cos\theta\nabla M.\nabla\right]\frac{p}{\rho_0 c_0^2} =$$

$$= -\frac{\partial}{\partial t}\{C\ \nabla.f + 2\cos\theta\nabla M.f\} + \frac{\partial^2}{\partial t_2}(C^2 q) \qquad (16)$$

where $M = U/c_\infty$ and $C = 1 - M\cos\theta$.

Dropping the overall $\partial/\partial t$ and dividing the resulting equation by C^{-3} leads, after the last two terms in the operator, as well as those related to the dipole source term, are grouped togheter, to the equivalent form

$$\frac{1}{\rho_0 c_0^2}\{\frac{\partial^2}{\partial t^2} - \nabla.(\frac{c_0^2}{C^2}\nabla)\}\ p = -\nabla.(\mathbf{f}\,C^{-2}) + C^{-1}\frac{\partial q}{\partial t} \qquad (17)$$

This is a most convenient form which evidences that, with regards to far field solutions, the problem of sound generation and propagation in a unidirectional shear flow is equivalent to the corresponding problem in a stationary medium with sound speed χ and density ξ given by

$$\chi = \frac{c_0}{1-M\cos\theta} \qquad \xi = \rho_0(1-M\cos\theta)^2 \qquad (18)$$

The right hand side of (17) can be interpreted as dipole and monopole distributions, fC^{-2} and $C^{-1}q$, per unit mass (or $\rho_0 f$ and $\rho_0 qC$ per unit volume). For $\mathbf{f} = -\nabla.T$, the corresponding source term

$$\nabla.(C^{-2}\nabla.T) = C^{-2}\,\nabla.\nabla.T + \nabla C^{-2}.\nabla.T = \nabla.\nabla.(TC^{-2}) - \nabla.(T.\nabla C^{-2}) \qquad (19)$$

can be writen as a sum of exact divergences, i.e., as a quadrupole and a dipole term. The intermediate expression in (19) was writen to emphasize the dependence between the terms that are seen in the source function of (1) for $\mathbf{f} = -\nabla.\mathbf{T}$, $q = 0$, since neither of them alone can be interpreted, in the equivalent problem (17–19), as a quadrupole or a dipole distribution (unless, of course, $\nabla C = 0$). Thus, if only the first term is considered, as in equation (10), since it corresponds to $C^{-2}\,\nabla.\nabla.\mathbf{T}$ which, from (19), can be writen as the sum of two physical sources minus $\nabla C^{-2}.\nabla.\mathbf{T}$, it follows that this last term is found responsible for the anomalous behaviour observed in [2].

The solution of (17), with f replaced by $\mathbf{f} - \nabla.\mathbf{T}$, can be represented as

$$p(\mathbf{x},t) = \iint S(\cos\theta,\mathbf{y},\tau)G(\mathbf{x},\mathbf{y},t-\tau)\, dV_{\mathbf{y}}\, d\tau \tag{20}$$

where the Green's function G is the solution of

$$\mathscr{L}(\mathbf{x},t)G(\mathbf{x},\mathbf{y},t-\tau) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau) \tag{21}$$

$\mathscr{L}$ being the operator in (18) and S, which can be split into $S = S_M + S_D + S_Q$ is given, after the space and time operators have been transferred to G, by

$$S_M = -\,C^{-1}\,q(\mathbf{y},\tau)\,\frac{\partial}{\partial\tau} \tag{22}$$

$$S_D = C^{-2}\,\mathbf{f}(\mathbf{y},\tau).\nabla \tag{23}$$

$$S_Q = C^{-2}\,\mathbf{T}(\mathbf{y},\tau):\nabla\nabla + \nabla C^{-2}.\mathbf{T}(\mathbf{y},\tau).\nabla \tag{24}$$

There is still further dependence between the two terms in (24) as is evidenced by the work of Tester and Morfey [2]. Since the resulting expressions are similar both for a round jet and for a plane shear layer, from here onward it will be supposed that U and c_0 can be writen as functions of a single transverse coordinate, x_t, meaning either the radial coordinate or x_2. Thus, the second transverse derivative of G can be writen, since G satisfies

$$\mathscr{L}(\mathbf{y},t)G(\mathbf{x},\mathbf{y},\, t-\tau) = 0 \tag{25}$$

as

$$\frac{\partial^2 G}{\partial y_t^{\,2}} = \{\frac{1}{\chi^2}\frac{\partial^2}{\partial t^2} - (\nabla^2 - \frac{\partial^2}{\partial y_t^{\,2}}) - \frac{1}{\chi^2}\,\nabla.(\chi^2\nabla)\}G \tag{26}$$

With the use of (26), the expression for $\nabla\nabla G$ can be split into

$$\nabla\nabla G = (\nabla\nabla)_0 G - \frac{1}{\chi^2}\,\nabla\chi^2.\nabla G\; \mathbf{e}_t\otimes\mathbf{e}_t \tag{27}$$

where $\mathbf{e}_t$ is the unit vector in the y_t direction and the operator $(\nabla\nabla)_0$, actually defined by (27), represents $\nabla\nabla G$ in a form suitable for an isothermal slug flow, when $\nabla\chi = 0$.

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Thus, SQ can be writen

$$SQ = C^{-2}\{T:(\nabla\nabla)_0 - T_{tt}\frac{\nabla c_0^2}{c_0^2}.\nabla + C^2\nabla C^{-2}.T.\nabla_1\} \quad (28)$$

where $\nabla_1 = \nabla - e_t\partial/\partial y_t$, what shows that, for volume acceleration sources, the contribution of $(dM/dy_t)T_{tt}$ is totally cancelled. Note that the terms multiplied by the gradients of c_0^2 and C^{-2} are not the same, what is due to the fact the dipoles in the original problem were described as sources per unit mass. For sources per unit *volume*, this difference would not occur.

The solutions for volume displacement dipoles and quadrupoles, $\mathbf{f}^*$ and $\mathbf{T}^*$, can be obtained from (22), for $q = D_0(-\nabla.\mathbf{f}^* + \nabla.\nabla.\mathbf{T}^*)/Dt$. The corresponding S^* are

$$SD^* = \mathbf{f}^*.\nabla\frac{\partial^2}{\partial t^2} \quad (29)$$

$$SQ^* = \mathbf{T}^*:\nabla\nabla\frac{\partial^2}{\partial t^2} = \{T:(\nabla\nabla)_0 - T_{tt}(\frac{\nabla c_0^2}{c_0^2} + C^2\nabla C^{-2}).\nabla\}\frac{\partial^2}{\partial t^2} \quad (30)$$

where $\partial^2/\partial t^2$ replaces $\partial^2/\partial\tau^2$ and equations (26) and (27) were used in the obtention of (30).

The comparison of (23) and (24) with (29) and (30) evidences two points: first, for volume displacement sources, the factor C^{-2} gives place to the operator $\partial^2/\partial t^2$. Since for a point source moving in the x_1 direction with constant velocity $c_\infty M_c$, $\partial^2G/\partial t^2$ will be replaced by $(1-M_c\cos\theta)^{-2}\partial^2G/\partial t^2$, this is seen as a minor difference for sources moving with the mean flow, the solution for moving dipole sources being practically equivalent. The other point concerns only the quadrupole solutions and regards the components that will be affected by the mean flow gradient, T_{tt} for the volume displacement source and those having only one index $_t$, for the volume acceleration source. As the efficiency of these terms is augmented to that of dipoles, the dominant quadrupole components in the formal solutions will depend also on the type of source.

This aspect brings to mind the relevance of mean shear amplification to the jet noise problem, previously (and most fully) discussed in [3]. Since it can be shown that, as in that case, dU/dy_t actually scales as $\partial/\partial t$, it turns out that the expected efficiency increase due to mean shear is only apparent, being inexistent for jet noise. Thus, for isothermal jets, all terms in (28) will present the same efficiency, no difference of character existing between the two groups of terms. This justifies the use of the simpler slug flow analogy, derived by Dowling, Ffowcs Williams and Goldstein [7], for the description of the noise of cold jets for any frequency limit. The dipole term due to gradients of c_0^2 (that is, of temperature), on the other hand, will be important for hot jets. But since in this case the explicit dipole $-c^{2\prime}\nabla\pi$ must also be considered as a source, a simple adjustment of its directivity can answer for the mean gradient term. It is interesting to note that the computations of [2] indicate that most of the dipole noise is due to the randomly orientated dipole, not to that related to the mean temperature gradient. Thus, since the source terms proportional to mean gradients do not actually introduce novel features in the solutions, the slug flow analogy can be used as a valid substitutive for Lilley's equation for the description of basic jet noise properties. As consequence of this, it follows that, although jet noise sources correspond physically to volume acceleration sources, as longs as the relationship between mean shear and time derivatives is recognized, no significant difference will be detected if the sources are modelled as volume aceleration or as volume displacement dipoles and quadrupoles.

6. CONCLUSION

The description of the source terms for Lilley's equation has been reviewed, it being shown that it is possible to write all these source terms as multipole sources in a shear flow, as well as to attach to each of them a clear physical meaning. It was also shown that the characterization of jet noise sources as of the volume diplacement type is not appropriate, having its origin in the neglect of second (and third) order source terms that do not have a physical interpretation of their own as such, which led to non–physical sources.

An analogy between Lilley's equation and the problem of sound generation and propagation in a stationary medium with non uniform sound speed has been established and explored for the discussion of properties of the solutions. The analysis shows that the main features of jet noise can be described by the simpler slug flow analogy, in which both the mean velocity and sound speed are constant within the shear layer.

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