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THE DESCRIPTION AND SIMULATION OF CORRELATED NON-GAUSSIAN NOISE

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INTRODUCTION

The fluctuations in a physical system, which cannot be described by the classical macroscopic phenomenology of mathematical physics, are of considerable interest, both as a source of information about the structure and dynamics of the system and as experimentally unavoidable noise which corrupts macroscopic measurements of the properties of the system. The Gaussian random process and its simple generalizations have long been recognised to be useful models for these fluctuations and the mathematical description of such processes has been stimulated and influenced strongly by insights derived from the physics of fluctuating systems and in particular from the phenomenon of Brownian motion[1]. However it has become increasingly apparent that the fluctuations in many systems are modelled more effectively by non-Gaussian noise processes. These non-Gaussian processes can be analysed both formally and by computer simulation, principally by methods again inspired by the study of Brownian motion. After introducing the main tools for this analysis, the Fokker Planck (F.P) equation and stochastic differential equations (s.d.e) [2] and illustrating their use by considering several simple, but non-trivial, non-Gaussian noise processes we will demonstrate how it is possible to describe and simulate the K distributed process, which has been recognised to be of considerable importance and widespread utility [3]. An extension of this model to include the effect of mixing noise of this type with a coherent signal will also be outlined.

FOKKER PLANCK AND STOCHASTIC DIFFERENTIAL EQUATIONS

The F.P equation is a linear partial differential equation obeyed by the probability density function (p.d.f) $P(x,t)$ of a random process χ, from which the probability that χ takes values between x and $x+dx$ at time t is given by $P(x,t)dx$. For a one dimensional process the F.P. equation may be written as

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x}\left(a(x)\ P(x,t)\right) + \frac{\partial^2}{\partial x^2}\left(b^2(x)\ P(x,t)\right) \tag{1}$$

The stationary statistical properties of χ are described by the time-independent solution $P(x,\infty)$ of (1) satisfying

$$\frac{\partial}{\partial t} P(x,\infty) = 0 \tag{2}$$

which may be shown to take the form

$$P(x,\infty) = \frac{C}{b^2(x)} \exp\left(\int^x dx' \frac{a(x')}{b^2(x')}\right) \tag{3}$$

where C is determinedby the requirement that $P(x,\infty)$ is normalised. As processes described by F.P equations are necessarily Markovian the temporally varying statistical properties of χ are determined by $p(x,t|x_0)$, the conditional probability that X take the value x a time t after having the value x_o. This may be obtained as the solution of (1) satisfying the initial condition

$$P(x,0|x_o) = \delta(x - x_o) \tag{4}$$

An alternative way in which a stochastic element may be introduced into the description of a process X is to propose an equation of motion for x which includes an explicit randomly varying term. Such a stochastic differential equation may be written as

$$\frac{dx}{dt} = a(x) + b(x)\, f(t) \tag{5}$$

where f(t) is the Gaussian white noise process. As long as this s.d.e is supplemented by the Ito interpretative calculus [2] (1) and (5) provide stochastically equivalent descriptions of the same process. The linearity of the F.P equation and its classical mathematical form make it a suitable vehicle for formal analyses; the equivalent s.d.e provides a convenient route to the simulation of the process by numerical integration using one of several algorithms described in the literature [4]. Finally we note that systems described by F.P.E and s.d.e in which b(x) is a constant are said to exhibit additive noise; when b(x) depends explicitly on x the noise is said to be multiplicative.

SOME EXAMPLES

We illustrate the foregoing discussion by a few examples presented in essentially tabular form; the qualitative properties of the noise processes will be evident from realisations generated by numerical simulation.

1) $a(x) = -\alpha x, \quad b(x) = 1.$ — Gaussian, Ornstein Uhlenbeck process

$$P(x,\infty) = \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \exp -\left(\frac{\alpha x^2}{2}\right)$$

Fig 1.

2) $a(x) = \beta - \alpha x, \ b(x) = x^{\frac{1}{2}}$ — Gamma process. $x > 0$ t natural boundary at origin

$$P(x,\infty) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}$$

Fig 2.

3) $a(x) = \beta - \alpha x, \ b(x) = x$ — Power low tail in distribution, note occasional large fluctuations $\alpha = -\frac{1}{2}$ - Levy stable distribution. Natural boundary at origin.

$$P(x,\infty) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \frac{e^{-\beta/x}}{x^{\alpha+2}}$$

Fig 3.

GAMMA AND RAYLEIGH PROCESSES AND MULTIPLICATIVE NOISE

So far nothing has been said about the origin of the F.P and s.d.e we have discussed. While the original s.d.e of Brownian motion were obtained by supplementing deterministic equations of motion by a random term this procedure, when invoked without the penetrating physical insight of Langevin, provides an uncontrolled and unreliable description of a physical system, particularly when multiplicative noise is involved. If, however, a F.P.E. can be deduced from either sound physical principles or the systematic reduction of a more detailed description of the system this will provide us with an immediate justification for its stochastically equivalent s.d.e. Here we consider two simple cases which are directly related to the description of the K-process. Jakeman [5] has shown that the coherent illumination of a collection of scatterers with a negative binomial distribution will, in the limit of a large mean number of scatterers, yield a K distributed intensity of scattered light. The negative binomial distribution can be established by the competing processes of birth, death and migration described by the rate equation

$$\frac{d\,P_N(t)}{dt} = \mu(N+1)P_{N+1}(t) - \Big((\lambda+\mu)N+\nu\Big)P_N(t) + \Big(\lambda(N-1)+\nu\Big)P_{N-1}(t) \tag{6}$$

$P_N(t)$ is the probability that N scatterers are illuminated at time t; λ, μ, ν characterise the uncorrelated processes of birth, death and migration respectively. By introducing the essentially continuous variable x through

$$N = \overline{N}\,x$$

where $\overline{N} = \nu/(\mu-\lambda)$ is the mean number of scatterers, scaling the time variable $(t \to t/\overline{N})$ and expanding $P_{N\pm1}(t)$ in Taylor series the following equation for $P(x,t) \equiv P_{\overline{N}x}(t\overline{N})$ can be derived

$$\frac{\partial P}{\partial t} = \lambda\frac{\partial^2}{\partial x^2}(xP) + \nu\frac{\partial}{\partial x}\Big((x-1)P\Big), \tag{7}$$

which is the F.P.E describing a gamma process with the stationary distribution

$$P(x,\infty) = \frac{\alpha^\alpha}{\Gamma(x)}\, e^{-\alpha x}\, x^{\alpha-1}, \quad \alpha = \frac{\nu}{\lambda}.$$

Thus this simple expansion procedure yields a F.P equation exhibiting multiplicative noise; viewed from the point of view of the s.d.e such a description could hardly be thought to be obvious. The second process implicit in Jakeman's model is that of coherent scattering in which the intensity of scattered light I is determined by the statistical properties of the phases of the electric fields scattered by a collection of illuminated objects. If the phases are assumed to be independent then an exponential distribution of scattered intensity results in the limit of a large number of scatterers. This is a special case of the gamma distribution and can be described by the F.P.E.

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial z}\Big((z-1)P\Big) + \frac{\partial^2}{\partial z^2}(zP) \tag{8}$$

where $z = I/\langle I\rangle$. This equation has the required stationary solution and also describes the temporal properties implicit in those of the phases in the conventional scattering model. Writing the electric field as

$$\varepsilon(t) = \sum_j a_j(t)\, e^{i\phi_j(t)}, \tag{9}$$

forming the intensity through

$$I = \varepsilon\,\varepsilon^* \tag{10}$$

and constructing the normalised correlation function $C_{n,m}(t) = \langle I^n(t)I^m(0)\rangle / \langle I\rangle^{n+m}$ gives, in the limit of a large number of scatterers

$$C_{n,m}(t) = n!m! \sum_{r=0}^{\min(n,m)} \frac{(n!m!)}{(n-r)!r!(m-r)!r!} |g_1(t)|^{2r}. \qquad (11)$$

The form of this result incorporates the assumed statistical properties of the phases $\langle \exp i(\phi_k(t) - \phi_\ell(t'))\rangle = g_1(t-t')\delta_{k\ell}$ and combinational factors arising from the products of sums of exponentials representing powers of the intensity. $C_{n,m}(t)$ may also be formed from the fundamental solution of (8) through

$$C_{n,m}(t) = \iint dz dz_o \, P(z,t|z_o) z^n z_o^m \, P(z_o,\infty). \qquad (12)$$

Expansion of $P(z,t|z_o)$ in Laguerre polynomials

$$P(z,t|z_o) = e^{-z} \sum L_r(z) L_r(z_o) \, e^{-rt} \qquad (13)$$

and use of the integral

$$\int_0^\infty dz \, z^n \, e^{-z} L_r(z) = (-1)^r \frac{(n!)^2}{(n-r)!r!} \quad n \geq r \qquad (14)$$

$$= 0 \quad n < r$$

demonstrate how (12) and ultimately the structure of the F.P operator (8) encode the combinational factors implicit in the assumed statistical properties of the phases $\phi_j(t)$. Thus the F.P formulation, couched in terms of the intensity alone, provides us with a compact and powerful description of the scattering process.

FOKKER-PLANCK DESCRIPTION OF THE K PROCESS

To construct a K process we consider a Rayleigh process z with a current mean value x, which is itself gamma distributed. The stationary joint distribution of x, z is then

$$P(z,x) = \frac{1}{\Gamma(\nu+1)} x^{\nu-1} e^{-x} e^{-z/x} ; \qquad (15)$$

the marginal distribution of z is a K distribution

$$P(z) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty dx \, x^{\nu-1} e^{-x} e^{-z/x} = \frac{2z^{\nu/2}}{\Gamma(\nu+1)} K_\nu(2\sqrt{z}) \qquad (16)$$

A F.P.E describing the dynamics of x, z and having (15) as its stationary solution is

$$\frac{\partial}{\partial t} P(x,z;t) = \mathcal{A}\left(\frac{\partial^2}{\partial x^2}(xP) + \frac{\partial}{\partial x}\left((x - \nu - \frac{z}{x})P\right)\right) + \mathcal{B}\left(\frac{\partial^2}{\partial z^2}(zP) + \frac{\partial}{\partial z}\left((\frac{z}{x} - 1)P\right)\right) \tag{17}$$

The equivalent s.d.e s are

$$\frac{dx}{dt} = \mathcal{A}\left(\nu - x + \frac{z}{x}\right) + \mathcal{A}^{\frac{1}{2}}x^{\frac{1}{2}}f_1(t)$$
$$\frac{dz}{dt} = \mathcal{B}\left(1 - \frac{z}{x}\right) + \mathcal{B}^{\frac{1}{2}}z^{\frac{1}{2}}f_2(t). \tag{18}$$

Both (17) and (18) are made up of components reminiscent of the gamma and Rayleigh processes, suitably coupled to yield (15) as a stationary solution. Some formal analysis of (17) is possible; in particular projection operator techniques can effect the separation of the intensity autocorrelation function into a slowly varying number fluctuation term and a rapidly decaying speckle term, reproducing Jakeman's generalised Seigert relation. The existence of the stationary solution (15) ensures the existence of a complete, orthonormal set of eigenfunctions of the F.P operator [2]. These however have not been found in closed analytic form. Numerical integration of the s.d.e provides a route to the simulation of K noise; simulated and experimentally measured K-noise are shown in Fig 4.

When K noise is mixed with a coherent signal of intensity a the resulting intensity has a distribution given by

$$P(z) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty x^{\nu-1} e^{-x} \exp\left(-\frac{(z+a)}{x}\right) I_0\left(\frac{2\sqrt{za}}{x}\right)dx \tag{19}$$

This may be regarded as a superposition of Rice processes with a gamma distributed noise power; I_n is a modified Bessel function. The F.P description of the process analogous to that of the K process is given by

$$\frac{\partial P}{\partial t} = \mathcal{A}\left(\frac{\partial^2}{\partial x^2}(xP) - \frac{\partial}{\partial x}\left(\left(\nu - x + \frac{a+z}{x} - \frac{2\sqrt{za}}{x}\frac{I_1\left(\frac{2\sqrt{za}}{x}\right)}{I_0\left(\frac{2\sqrt{za}}{x}\right)}\right)P\right)\right) + \mathcal{B}\left(\frac{\partial^2}{\partial z^2}(zP) - \frac{\partial}{\partial z}\left(\left(1 - \frac{z}{x} + \frac{2\sqrt{az}}{x}\frac{I_1\left(\frac{2\sqrt{za}}{x}\right)}{I_0\left(\frac{2\sqrt{za}}{x}\right)}\right)P\right)\right) \tag{20}$$

while the equivalent pair of s.d.e is

$$\frac{dx}{dt} = \mathcal{A}\left(\nu - x + \frac{a+z}{x} - \frac{2\sqrt{za}}{x}\frac{I_1(2\sqrt{za}/x)}{I_o(2\sqrt{za}/x)}\right) + \mathcal{A}^{\frac{1}{2}}x^{\frac{1}{2}}f_1(t)$$

$$\frac{dz}{dt} = \mathcal{B}\left(1 - \frac{z}{x} + \frac{2\sqrt{za}}{x}\frac{I_1(2\sqrt{za}/x)}{I_o(2\sqrt{za}/x)}\right) + \mathcal{B}^{\frac{1}{2}}z^{\frac{1}{2}}f_2(t) \qquad (21)$$

(20), (21) reduce to (17),(18) when a = O. Relatively little formal analysis of these complex equations is possible although we again have the stationary solution, a complete orthonomial set of (unknown) eigenfunctions and a separation into fast and slow components when $\mathcal{B} \gg \mathcal{A}$. A simulated sample of this homodyned K noise is shown in Fig 5.

CONCLUDING REMARKS

It has been shown that the Fokker Planck and stochastic differential equation formalisms provide a useful context within which to analyse and simulate non-Gaussian noise processes. In particular it has been possible to accommodate the K process within this framework.

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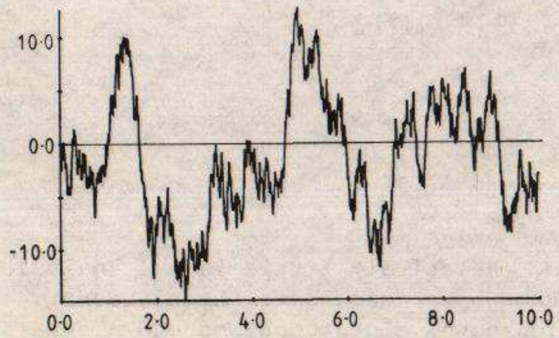


Fig.1. The Ornstein-Uhlenbeck process

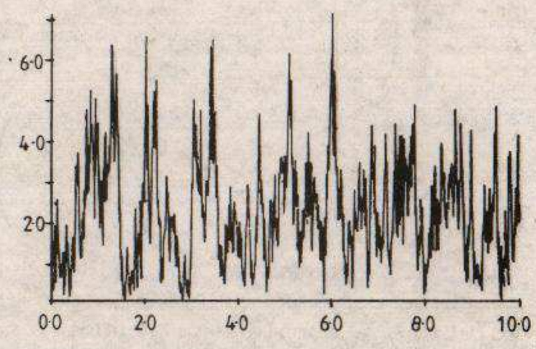


Fig 2. The gamma process

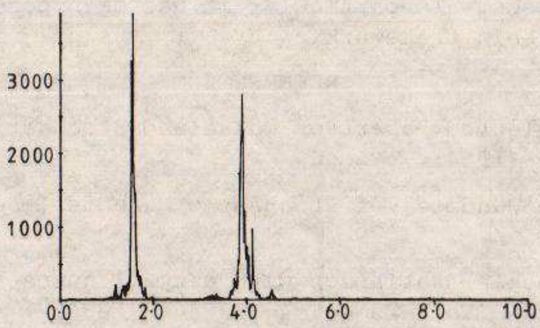
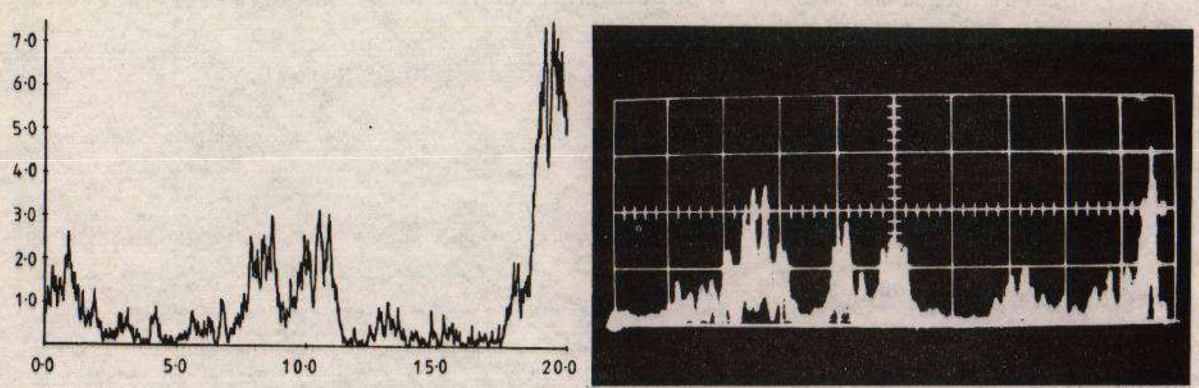


Fig 3. The Levy stable process



(a) (b)

Fig 4.

The K process (a) simulated, and (b) experimentally measured