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THE STATISTICAL PROPERTIES OF RANDOM FREQUENCY RESPONSE FUNCTIONS

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In many industrial situations the vibro-acoustic response of an engineering system is sensitive to manufacturing imperfections. Such imperfections are uncertain, or random, and this implies that any given system is necessarily drawn from an ensemble of possible systems. In some cases this is a "real" ensemble, as in the case of a collection of motor vehicles produced on a production line. In other cases the ensemble may be "virtual" in the sense that only one system is actually manufactured (for example, a satellite), although the properties of that one system can be considered to be drawn from a hypothetical ensemble of possibilities. In all cases the effect of the uncertainty should be considered in assessing the performance of the system: it is not sufficient for a nominally perfect system to meet the design requirements. Rather, there should be an acceptably low probability that the random system will not meet the design requirements. Any assessment of the random response of the system clearly requires knowledge of the statistics of the response, and in vibro-acoustics this normally involves a consideration of the statistics of a frequency response function (FRF). It initially appears to be a very challenging task to predict the statistical properties of FRFs, particularly when the underlying system uncertainties may be ill-defined or unknown. However, it is shown here that under broad conditions a universal behaviour arises, and random FRFs have very remarkable properties. The key result is that the average value of any function of an FRF is equal to the function evaluated at the average value of the FRF. This is known as the AE property in nuclear physics, and it is shown here that the result also applies in vibro-acoustics and has enormous utility in response prediction.
Keywords: vibro-acoustics, random systems.

1. Introduction

The vibration response of an engineering system can be extremely sensitive to manufacturing variability, to the extent that the frequency response functions (FRFs) of the system can be considered to be random over an ensemble of manufactured items. Ideally the statistical properties of the FRFs should be calculated at the design stage to ensure that the reliability and performance of the system will meet the required targets. In general, the statistics of the FRFs will depend on the statistical distribution of the manufacturing uncertainties, and it can be an extremely difficult task to (i) statistically quantify the manufacturing uncertainties, and (ii) propagate these uncertainties through a computational model of the system to yield the response statistics. However, if the system is sufficiently random then a number of "universal" statistical laws may be applicable, making it possible to avoid much of the difficulty involved in (i) and (ii). For example, it is known that the higher natural frequencies and mode shapes of a random system often conform to the statistics of the Gaussian Orthogonal Ensemble (GOE) [1,2], and this fact can be used in conjunction with energy flow models (in particular, Statistical Energy Analysis [3]) to yield the mean and variance of the vibrational energies of the various system components [4]. The concern of the present paper is not with the system energy distribution, but rather with the statistical properties of the complex FRFs, and in particular, the aim is to explore possible universal properties of these functions. Previous work in this area

includes that of Lyon regarding the variance of the modulus of a FRF [5] and the statistics of the phase of a FRF [6], and Skudrzyk [7] and Cremer and Heckl [8] regarding the mean value of a complex FRF. As explained below, these investigations are extended here by considering the potential applicability of a powerful statistical property of a FRF known as the analyticity-ergodic requirement.

A fundamental statistical property of a random FRF is the average value, and the average considered might be either a frequency average or an ensemble average. A frequency average can be taken over a rectangular frequency window, or for mathematical convenience a more complex window such as the Lorentzian weighting function can be used [9]. The Lorentzian weighting function, yielding the Lorentzian average, has the same shape as the Cauchy distribution [10] and is a bell-shaped curve with specified centre frequency and half-power bandwidth; the function has a single pole in the lower complex half-plane (and also a single pole in the upper half-plane), and because of this the Lorentzian average has remarkable properties when applied to a causal FRF (i.e. a function that is analytic in the lower half-plane). If the symbol $< x >$ is used to represent the Lorentzian average of x, then under loose restrictions it is found that $< f(H) >= f(< H >)$ for a function f of a causal FRF H [9]. In considering a random FRF in an engineering context, the ensemble statistics of the function are of more concern than Lorentzian frequency averages taken on a single realization, and it is therefore of significant interest to consider whether the ensemble average will have the same properties as the frequency average. This issue has been addressed in nuclear physics literature in the context of random scattering matrices: it is generally assumed that H is an ergodic random function [11], so that the frequency and ensemble averages are equal (for a sufficiently wide frequency averaging window) meaning that the ensemble average has exactly the same properties as the Lorentzian average [9]. The condition $< f(H) >= f(< H >)$, when applied to ensemble averages, is known as the analyticity-ergodicity (AE) requirement, and Mello et. al. [9] have used this requirement in conjunction with the principle of maximum entropy to yield the probability density function of a scattering matrix. More generally, the literature on random scattering matrices, with potential application to vibrational FRFs, is very extensive: references [12-14] are examples of review papers, and a very recent contribution [15] has derived exact results for the probability density function of the real or imaginary part of an off-diagonal element of a scattering matrix (analogous to a cross-admittance in vibration theory). The results reported in reference [15] are based on the assumption that the scattering object has either Gaussian Orthogonal Ensemble (GOE) or Gaussian Unitary Ensemble (GUE) statistical properties, and the AE requirement is not considered explicitly. The analysis of reference [15] is not immediately applicable to engineering systems, and despite the extent of the literature on random scattering matrices it is not clear whether the AE requirement will apply to vibrational FRFs, as discussed in what follows.

As noted above, the AE requirement is justified in nuclear physics by arguing that the properties of the Lorentzian average also hold for the ensemble average. For this to be a valid argument H must ergodic, and this implies that H must also be a stationary random function [11], or at least approximately stationary over the window of the frequency averaging function. While this condition is likely to be met in nuclear physics, the condition is much less likely to apply in vibration and acoustics. It is therefore not possible to draw conclusions regarding the properties of the ensemble averaging process by appealing to ergodicity arguments. Nonetheless, simulations of random vibrational systems indicate a strong tendency for the ensemble average E[] to have the same properties as the Lorentzian average, so that $\mathrm{E}[f(H)] = f(\mathrm{E}[H])$, and this raises the question of whether this is in some way a universal result, and if so, what conditions must apply for the result to hold. This issue is addressed in the present paper.

The properties of the Lorentzian average of an FRF are considered in Section 2. The properties of the ensemble average are then explored in Section 3 by using random point process theory [16] to model the system randomness. It is found that the ensemble average will have the same properties as the Lorentzian average providing the kernel functions that appear in the random point process theory are stationary. The stationarity of the kernel functions does not imply that the FRF is station-

ary, and the condition is actually quite weak: the kernels need to be either locally stationary, or stationary under an unfolding transformation [1,17], and there is evidence to suggest that these conditions are met for highly random vibratory systems. An example application is considered in Section 4, and a summary of the findings of the work is then given in Section 5. *The present paper is an abbreviated version of a much more detailed analysis contained in reference* [18].

2. The Lorentzian frequency average

The Lorentzian average of a function of frequency ω, $H(\omega)$ say, is defined as a weighted integral in the form

$$\left\langle H(\omega)\middle|\omega_0,\Gamma\right\rangle = \int_{-\infty}^{\infty} H(\omega)W(\omega,\omega_0,\Gamma)\mathrm{d}\omega \Big/ \int_{-\infty}^{\infty} W(\omega,\omega_0,\Gamma)\mathrm{d}\omega, \tag{1}$$

where the Lorentzian weighting function W is given by

$$W(\omega,\omega_0,\Gamma) = \left[(\omega-\omega_0)^2+\Gamma^2\right]^{-1}. \tag{2}$$

The parameter ω_0 locates the peak of the weighting function, while the parameter Γ determines the width of the function: the half-power bandwidth (i.e. the width of the function at half the peak amplitude) is given by 2Γ. If $H(\omega)$ represents a frequency response function, then the associated impulse response function $h(t)$ is given by the Fourier transform relation

$$h(t) = \int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\mathrm{d}\omega. \tag{3}$$

If the impulse response is causal then by definition $h(t)$ is zero for $t<0$ and this implies that $H(\omega)$ is analytic in the lower half-plane. The integrand in the numerator of Eq. (1) therefore has a single pole in the lower half-plane, located at $\omega=\omega_0-i\Gamma$, and contour integration around a contour enclosing this half-plane yields

$$\left\langle H(\omega)\middle|\omega_0,\Gamma\right\rangle = H(\omega_0-i\Gamma). \tag{4}$$

It can similarly be demonstrated that the Lorentzian average of any causal function of $H(\omega)$ can be expressed as

$$\left\langle f(H)\middle|\omega_0,\Gamma\right\rangle = f[H(\omega_0-i\Gamma)], \tag{5}$$

from which it follows that

$$\left\langle f(H)\middle|\omega_0,\Gamma\right\rangle = f\left(\left\langle H\middle|\omega_0,\Gamma\right\rangle\right). \tag{6}$$

Thus the Lorentzian average of a function of H is equal to the value of the function when evaluated at the Lorentzian average of H. At first sight Eq. (6) is a very unexpected result, since it implies, for example

$$\left\langle H^2(\omega)\middle|\omega_0,\Gamma\right\rangle = \left\langle H(\omega)\middle|\omega_0,\Gamma\right\rangle^2. \tag{7}$$

Were $H(\omega)$ real then Eq. (7) would imply that the variance of the function along the frequency axis is zero, and therefore the function has a constant value. However, it should be recalled that in general $H(\omega)$ is a causal complex frequency response function; the real and imaginary parts of a causal function are related by the Kramers-Kronig relation [19], and it is this dependence between the real and imaginary parts that allows Eq. (7) to apply even when the function is non-constant.

It is interesting to consider whether Eq. (6) might apply to the *ensemble* average of a random frequency response function at a single specified frequency ω, and this issue is addressed in the following section.

3. The ensemble statistics of causal functions

This section will consider the statistical properties of a set of causal frequency response functions $\{H_1(\omega),\ H_2(\omega),...,\ H_M(\omega)\}$ which all have the same set of simple poles. Each function can be expanded in terms of poles ω_j and residues a_j in the form

$$H(\omega)=\sum_j\left\{\frac{a_j}{\omega-\omega_j}-\frac{a_j^*}{\omega+\omega_j^*}\right\}\approx\sum_j\frac{a_j}{\omega-\omega_j}, \tag{8}$$

which can be generalised slightly to read

$$H_r(\omega)=\sum_j a_j^{(r)}h_r(\omega,\omega-\omega_j),\qquad r=1,2,...,M\ , \tag{9}$$

where the function $h_r(\omega,\omega-\omega_j)$ has a simple pole at $\omega=\omega_j$. The poles lie in the right upper half-plane, and they can be written in the form

$$\omega_j=z_j\exp(i\phi_j),\qquad z_j=\left|\omega_j\right|,\qquad \phi_j=\arg(\omega_j)\,. \tag{10}$$

If the function $u_r(\omega,\phi_j,z_j)$ is defined so that

$$u_r(\omega,\phi_j,z_j)\equiv h_r(\omega,\omega-z_je^{i\phi_j})\,, \tag{11}$$

then Eq. (9) can be written as

$$H_r(\omega)=\sum_j a_j^{(r)}u_r(\omega,\phi_j,z_j)\,. \tag{12}$$

For future reference it can be noted that when viewed as a function of z_j, at fixed ω and ϕ_j, the function $u_r(\omega,\phi_j,z_j)$ will have a simple pole in the right *lower* half-plane.

The joint statistics of the frequency response functions can be explored by considering the characteristic functional [16,20], which is defined as

$$\varphi[\theta_1(\omega),\theta_2(\omega),...,\theta_M(\omega)]=\mathrm{E}\left[\exp\left\{i\sum_{r=1}^{M}\int_{-\infty}^{\infty}H_r(\omega)\theta_r(\omega)\mathrm{d}\omega\right\}\right]. \tag{13}$$

The characteristic functional can be expressed in terms of the joint cumulants of the frequency response functions in the form [16,19]

$$\varphi=\exp\left\{i\sum_{r=1}^{M}\int_{-\infty}^{\infty}\kappa_1[H_r(\omega)]\theta_r(\omega)\mathrm{d}\omega+\frac{i^2}{2}\sum_{r=1}^{M}\sum_{s=1}^{M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\kappa_2[H_r(\omega_1'),H_s(\omega_2')]\theta_r(\omega_1')\theta_s(\omega_2')\mathrm{d}\omega_1'\mathrm{d}\omega_2'+...\right\} \tag{14}$$

where $\kappa_n[]$ represents the *n*th joint cumulant: for example, $\kappa_1[H_r(\omega)]$ is the mean value of $H_r(\omega)$ and $\kappa_2[H_r(\omega_1'),H_s(\omega_2')]$ is the covariance of the two functions H_r and H_s when evaluated at the frequencies ω_1' and ω_2' respectively. If the quantities z_j are taken to constitute a random point process and the phase quantities ϕ_j are (initially) taken to be deterministic, then the joint cumulants that appear in Eq. (14) can be written in the form [16]

$$\begin{aligned}\kappa_\alpha[H_{n_1}(\omega_1'),H_{n_2}(\omega_2'),...,H_{n_\alpha}(\omega_\alpha')]=&\int_0^\infty\int_0^\infty...\int_0^\infty k^{(\alpha)}_{n_1n_2...n_\alpha}(z_1,z_2,...,z_\alpha,\mathbf{a})\times\\&u_{n_1}(\omega_1',\phi_1,z_1)u_{n_2}(\omega_2',\phi_2,z_2)...u_{n_\alpha}(\omega_\alpha',\phi_\alpha,z_\alpha)\mathrm{d}z_1\mathrm{d}z_2...\mathrm{d}z_\alpha,\end{aligned} \tag{15}$$

where the kernel functions $k^{(\alpha)}_{n_1n_2...n_\alpha}(z_1,z_2,...,z_\alpha,\mathbf{a})$ are determined by the statistical properties of the random point process and the statistics of the residues $a_j^{(r)}$. The residues are assumed to have statistical properties that are independent of the index *j*, and in what follows, for ease of notation, a quantity such as $\mathrm{E}[a_j^{(r)}]$ is abbreviated to $\mathrm{E}[a^{(r)}]$. This assumption amounts to considering the residues $a_j^{(r)}$

to be at least locally stationary over the range of modes that contribute significantly to Eq. (12). Following Stratonovich [16], the first three kernel functions that appear in Eq. (30) can be written as

$$k_{n_1}^{(1)}(z_1,\mathbf{a}) = \mathrm{E}[a^{(n_1)}]g_1(z_1), \tag{16}$$

$$k_{n_1n_2}^{(2)}(z_1,z_2,\mathbf{a}) = \mathrm{E}[a^{(n_1)}a^{(n_2)}]g_1(z_1)\delta(\mathrm{z}_1-z_2) + \mathrm{E}[a^{(n_1)}]\mathrm{E}[a^{(n_2)}]g_2(z_1,z_2), \tag{17}$$

$$\begin{aligned} k_{n_1n_2n_3}^{(3)}(z_1,z_2,z_3,\mathbf{a}) &= \mathrm{E}[a^{(n_1)}a^{(n_2)}a^{(n_3)}]g_1(z_1)\delta(\mathrm{z}_1-z_2)\delta(\mathrm{z}_2-z_3) \\ &\quad + 3\left\{\mathrm{E}[a^{(n_1)}a^{(n_2)}]\mathrm{E}[a^{(n_3)}]g_2(z_1,z_3)\delta(\mathrm{z}_1-z_2)\right\}_s \\ &\quad + \mathrm{E}[a^{(n_1)}]\mathrm{E}[a^{(n_2)}]\mathrm{E}[a^{(n_3)}]g_3(z_1,z_2,z_3), \end{aligned} \tag{18}$$

and the higher order kernels follow a similar pattern. In Eq. (18) the notation $\{\}_s$ denotes an average taken over the distinct permutations of the arguments, and the functions g_n are referred to by Stratonovich [16] as the correlation functions of the point process. The same functions are referred to by Lin [20] as the cumulant functions, and in random matrix theory [17] (with a very minor change in definition) the functions are referred to as the *n*-level cluster functions. Whatever terminology is adopted, the functions are dependent on the statistical distribution of the poles of the system, which is in turn determined by the ensemble statistics of the system properties. It can be noted that: (i) in all cases the function g_1 corresponds to the modal density of the points, (ii) for a Poisson point process, $g_n = 0$ for $n>1$, (iii) for a point process conforming to the Gaussian Orthogonal Ensemble, the functions g_n are available from random matrix theory [17]. Now it can be assumed that

$$u_r(-\omega,\phi_j,z_j) \ll u_r(\omega,\phi_j,z_j), \tag{19}$$

so that the integrals in Eq. (30) can be extended to an infinite range to yield

$$\begin{aligned} \kappa_\alpha[H_{n_1}(\omega_1'),H_{n_2}(\omega_2'),...,H_{n_\alpha}(\omega_\alpha')] \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} k_{n_1n_2...n_\alpha}^{(\alpha)}(z_1,z_2,...,z_\alpha,\mathbf{a})\times \\ u_{n_1}(\omega_1',\phi_1,z_1)u_{n_2}(\omega_2',\phi_2,z_2)...u_{n_\alpha}(\omega_\alpha',\phi_\alpha,z_\alpha)\mathrm{d}z_1\mathrm{d}z_2...\mathrm{d}z_\alpha. \end{aligned} \tag{20}$$

If the random point process is now taken to be stationary, then the kernel functions have the property

$$k_{n_1n_2...n_\alpha}^{(\alpha)}(z_1,z_2,...,z_\alpha,\mathbf{a}) = k_{n_1n_2...n_\alpha}'^{(\alpha)}(z_2-z_1,...,z_\alpha-z_1,\mathbf{a}), \tag{21}$$

where $k_{n_1n_2...n_\alpha}'^{(\alpha)}$ is a modified kernel involving only the separation of the various poles. The joint cumulants of the frequency response functions can then be written in the form

$$\begin{aligned} \kappa_\alpha[H_{n_1}(\omega_1'),H_{n_2}(\omega_2'),...,H_{n_\alpha}(\omega_\alpha')] \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} k_{n_1n_2...n_\alpha}'^{(\alpha)}(y_1,y_2,...,y_{\alpha-1},\mathbf{a})\times \\ \left\{\int_{-\infty}^{\infty} u_{n_1}(\omega_1',\phi_1,z_1)u_{n_2}(\omega_2',\phi_2,y_1+z_1)...u_{n_\alpha}(\omega_\alpha',\phi_\alpha,y_{\alpha-1}+z_1)\mathrm{d}z_1\right\}\mathrm{d}y_1\mathrm{d}y_2...\mathrm{d}y_{\alpha-1}. \end{aligned} \tag{22}$$

It is important to note that Eq. (21) is a statement regarding the stationarity of the random point process, rather than the stationarity of $H_r(\omega)$: depending on the nature of $u_r(\omega,\phi_j,z_j)$, a stationary point process may lead to a non-stationary frequency response function $H_r(\omega)$. Regarding the stationarity of the point process, it is known that the modal density $n(z) \equiv g_1(z)$ of a number of common structural or acoustic components is not constant with frequency, and strictly Eq. (21) does not apply to such cases. The modal density of a bending beam, for example, is proportional to $z^{-1/2}$, while the modal density of an acoustic volume is proportional to z^2 [3,8]. However, there are two ways in which the use of Eq. (21) can be justified even for components of this type: (1) if the frequency of interest is sufficiently high, then the modal density can be considered to be locally constant over the modes that significantly contribute to the system response, and the same approximation can be applied to the higher order kernel functions; (2) the process of *unfolding* can be applied, in which the variable z is transformed to a new variable (the number count N) which has a constant modal density

– the transformed kernels are then found to be stationary to a remarkable degree for many systems [17]. If Eqs. (21) and (22) are accepted, then progress can be made by noting that: (i) each function $u_{n_j}(\omega_j', \phi_j, y_{j-1} + z_1)$ has a single pole in the complex z_1-plane and, following the discussion below Eq. (12), this pole lies in the lower half-plane, (ii) each function $u_{n_j}(\omega_j', \phi_j, y_{j-1} + z_1)$ is analytic in the upper half-plane, (iii) the integral over z_1 around a half-circle in the upper half-plane, centred at the origin and of infinite radius, is zero providing $r > 1$, (iv) for r=1, the integral around the half-circle has a finite, non-zero, value. By performing the integral over z_1 around a contour enclosing the upper-half plane it then follows from Eq. (22) that

$$\kappa_\alpha[H_{n_1}(\omega_1'), H_{n_2}(\omega_2'), ..., H_{n_\alpha}(\omega_\alpha')] = 0 \qquad \alpha > 1. \tag{23}$$

$$\kappa_1[H_r(\omega)] = \mathrm{E}[H_r(\omega)] \neq 0. \tag{24}$$

It follows immediately from this result and Eqs. (13) and (14) that

$$\mathrm{E}[H_{n_1}(\omega_1')H_{n_2}(\omega_2')...H_{n_\alpha}(\omega_\alpha')] = \mathrm{E}[H_{n_1}(\omega_1')]\mathrm{E}[H_{n_2}(\omega_2')]...\mathrm{E}[H_{n_\alpha}(\omega_\alpha')], \tag{25}$$

$$\mathrm{E}[H_r^s(\omega)] = \mathrm{E}[H_r(\omega)]^s. \tag{26}$$

This implies that the property noted previously for the Lorentzian average in Eq. (6) also applies to the ensemble average if the function f is a polynomial function or has a convergent Taylor series expansion, so that

$$\mathrm{E}[f(H)] = f(\mathrm{E}[H]). \tag{27}$$

These results imply that the Lorentzian and ensemble averages share the same properties, even though the results yielded by the two averaging processes may differ. The validity of Eq. (27) is explored numerically in the following section.

4. Numerical example

To illustrate the foregoing theory, an example system is considered which consists of a simply supported flat plate which is randomised by the addition of a number of small masses that are attached in random locations to generate an ensemble of systems. The out-of-plane motion of the plate at two locations $\mathbf{x}_1$ and $\mathbf{x}_2$ is investigated: the response at these locations (q_1 and q_2 say) to applied forces F_1 and F_2 of frequency ω can be written in the form

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \mathbf{H}(\omega) \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{28}$$

where the admittance matrix **H** is given by

$$\mathbf{H}(\omega) = \sum_j \left\{ \left(\frac{1}{\omega_{\mathrm{n}j}^2 - \omega^2 + 2i\beta_j\omega_{\mathrm{n}j}\omega} \right) \begin{pmatrix} \varphi_j^2(\mathbf{x}_1) & \varphi_j(\mathbf{x}_1)\varphi_j(\mathbf{x}_2) \\ \varphi_j(\mathbf{x}_1)\varphi_j(\mathbf{x}_2) & \varphi_j^2(\mathbf{x}_2) \end{pmatrix} \right\}. \tag{29}$$

Here $\omega_{\mathrm{n}j}$ is the jth natural frequency of the plate, φ_j is the jth (mass normalized) mode shape, β_j is the associated damping ratio, and proportional damping has been assumed. The modal parameters are random due to the presence of the small masses, and they can be computed by formulating the equations of motion of the plate using the Lagrange-Rayleigh-Ritz method, with the modes of the bare plate employed as trial functions. The dynamic stiffness matrix of the plate is given by

$$\mathbf{D}(\omega) = [\mathbf{H}(\omega)]^{-1}. \tag{30}$$

Numerical simulations of the random matrices **H** and **D** can be used to test various findings of the foregoing analysis. To this end a steel rectangular plate of length $l_1 = 0.9\mathrm{m}$, width $l_2 = 0.7\mathrm{m}$ and thickness 2mm is considered. The Young's modulus is $2\times10^{11}\,\mathrm{N/m^2}$, the density is $7800\ \mathrm{kg/m^3}$, the Poisson ratio is 0.3, and the damping factor is set at $\beta_j = 0.015$. The two ports are located at $\mathbf{x}_1 = (0.3l_1\ \ 0.43l_2)$ and $\mathbf{x}_2 = (0.23l_1\ \ 0.71l_2)$. The randomization is realized by the addition of 10 point

masses, randomly located on the plate with a uniform distribution, with each mass having 2% of the mass of the bare plate. In the results that follow, unless otherwise stated, averages have been taken over an ensemble of 2000 realizations. The modal density of the bare plate is $n(\omega) = 0.0164$ modes/rad/s, which means that the average modal spacing is 61.13 rad/s. The frequency range considered (0 to 6000 rad/s) covers approximately 100 resonant frequencies, and over this frequency range the modal overlap factor $m = 2\beta_j \omega_{nj} n$ varies linearly from 0 to 3.

Three realisations of the frequency response function $H_{11}(\omega)$ are shown in Fig. 1, together with the analytical result for the ensemble average value, which is given by $\mathrm{E}[H_{11}(\omega)] = -i\pi n / (2\omega M_p)$, where M_p is the mass of the bare plate [8]. It can be seen that the randomization approach has produced a great deal of variability in the frequency response function.

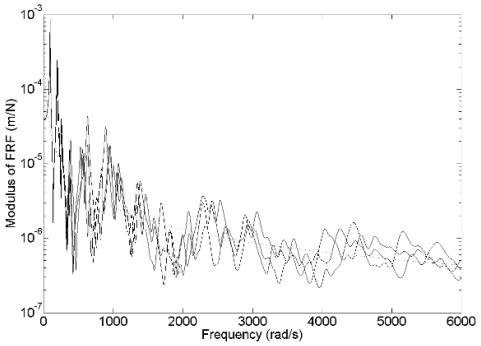

Figure 1: Three realisations of the response

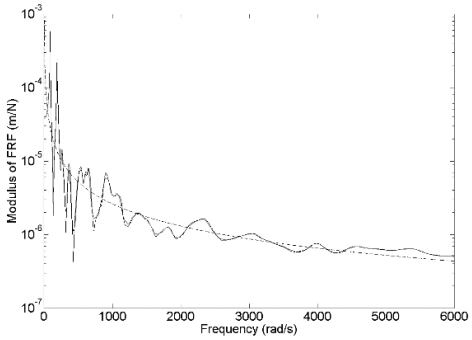

Figure 2: The average mobility

In Fig. 2 the validity of Eq. (27) is explored for the particular case

$$f(H_{11}) = H_{11}^{-1} \quad \Rightarrow \quad \mathrm{E}[H_{11}^{-1}] = \mathrm{E}[H_{11}]^{-1}. \tag{31}$$

The two ensemble average results shown in the figure (representing each side of the equation) show strong validation of Eq. (31) .

The results shown in Fig. 3 relate to the off-diagonal component D_{12} of the dynamic stiffness

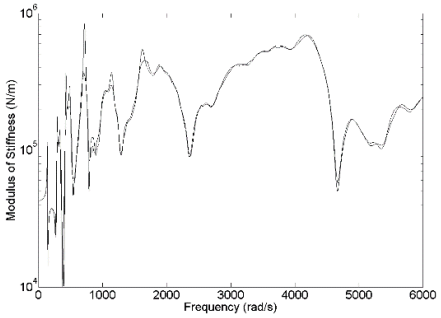

Figure 3: The average coupling stiffness

matrix, where the concern is with the validity of the following expression:

$$\mathrm{E}[f(\mathbf{H})] = \mathrm{E}\left[\frac{-H_{12}}{H_{11}H_{22} - H_{12}H_{21}}\right] = \frac{-\mathrm{E}[H_{21}]}{\mathrm{E}[H_{11}]\mathrm{E}[H_{22}] - \mathrm{E}[H_{12}]\mathrm{E}[H_{21}]}\ . \tag{32}$$

The numerical results confirm the validity of Eq. (32), despite the highly non-stationary character of the expectation (the left hand right sides of Eq. (32) are each plotted in the figure).

5. Conclusions

In brief, it has been shown that Eq. (27) is a remarkable property of random frequency response functions. Further details and examples are given in reference [18].

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