STUDY OF NONLINEAR PROPAGATION OF NOISE USING HIGHER-ORDER SPECTRAL ANALYSIS

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The report presents the results of numerical simulation of the evolution of intense acoustic noise in nondispersive media. The basic equation describing the propagation of nonlinear waves in nondispersive media is the Burgers equation (BE). It is known that during noise propagation in a nonlinear medium, changes occur to the spectra as well as the higher statistical characteristics of the wave (multipoint probability distribution, higher moments, and cumulants). The spectral method of the numerical solution of the BE allowed us to determine the range and the wave profile at different distances from the source (both before the formation of the shock front, and after it). It is very important that this scheme of numerical solution of the BE enables taking into account the geometry of the initial conditions, i.e., it is suitable for the description of cylindrical and spherical waves. However, the processing of acoustic signals through higher-order spectral analysis allows us to learn about the properties of the signal much more than the traditional spectral analysis. Higher-order spectral analysis (bispectral analysis) in signal processing tasks can save the information about the phase spectrum of the input signal, and therefore, it is possible to restore a priori unknown waveforms. The possibility of recovering the input spectrum from the measured spectrum and bispectrum at the output of the nonlinear medium is discussed. This model is a virtual instrument which can solve some diagnosis problems using spectral and higher-order spectral processing tools. All this suggests the possibility of using this scheme to obtain numerical solutions for inverse problems of nonlinear acoustics.

Keywords: nonlinear acoustics, bispectrum, Burgers equation

1. Introduction

At the end of the 1930s, the Dutch scientist J.M. Burgers [1] introduced a one-dimensional model for pressureless gas dynamics. This model is now known as the Burgers equation and it has not only the same type of hydrodynamical quadratic nonlinearity as the Navier–Stokes equation that is balanced by a diffusive term, but also similar invariances and conservation laws [2-4]. One of the most import applications of the Burgers equation both for regular and random fields is nonlinear acoustics [5,6], including investigation of intensive aircraft noise [7,8]. The evolution of high-order spectral functions of random waves propagating in nonlinear medium and application of higher-order spectra in problems of strong acoustic noise diagnosis was also investigated [9-12]. Retrieval of radiated-signal parameters from the measured acoustic field far from the source is one of the important problems in nonlinear acoustics [13]. In general, the problem is to restore the shape of the emitted signal or the original spectrum. In this paper, we discuss the possibility of restoring the parameters of regular and noise signals in the case of zero and vanishingly low viscosity. We show that one of the effective methods to restore the original spectrum is to use bispectra.

2. The inverse problem of nonlinear acoustics in the time-distance domain.

In this part, we consider the inverse problem of nonlinear acoustics in the time-distance domain: assuming that we know the velocity field $v(t,z)$ at some distance from the input, we try to reconstruct the field $v_0(t)$ at the input ($z=0$). The propagation of an intense plane acoustic wave can be described within the framework of the Burgers equation [5]

$$\frac{\partial v}{\partial z} - v\frac{\partial v}{\partial t} = \Gamma\frac{\partial^2 v}{\partial t^2} \tag{1}$$

When writing (1) we used the dimensionless variables

$$v = \frac{p}{p_0}, z = \frac{x}{l_{SH}}, t = \omega_0 \tau. \tag{2}$$

Here p is the acoustic pressure, x is the coordinate along the beam axis, t is the time in a coordinate system moving with the wave speed of sound c_0. Variables (2) are normalized to the peak value of pressure p_0, the characteristic frequency of the wave ω_0, and nonlinear length l_{SH} - which is the distance at which a discontinuity in the plane wave that is harmonic at the input is formed:

$$l_{SH} = \frac{c_0^3 \rho_0}{\varepsilon\, \omega_*\, p_0},\; l_{DISS} = \frac{2c_0^3 \rho_0}{b\, \omega_*^{\,2}}. \tag{3}$$

The ratio of these lengths forms a dimensionless number $\Gamma = l_{SH} / l_{ISS}$ - the inverse acoustic Reynolds number (the number of Goldberg). At zero viscosity, the Burgers equation (1) becomes the Riemann equation which is also known as the Hopf equation, or the equation of a simple wave

$$\frac{\partial v}{\partial z} - v\frac{\partial v}{\partial t} = 0 \tag{4}$$

Equation (4) is equivalent to the characteristics of the system, the solution of which

$$V(\tau,z) = v_0(\tau), \qquad T(\tau,z) = \tau - z v_0(\tau) \tag{5}$$

is a solution in parametric representation and allows us to write the solution of the Riemann equation in implicit form: $v = v_0(t + zv)$. To find the field $v(t,z)$ at time t in at a distance z from a source it is necessary to solve a nonlinear equation

$$t = \tau_* - z v_0(\tau_*), \qquad \tau_* = \tau_*(t,z) \tag{6}$$

and then the velocity field can be represented as

$$v(t,z) = -\frac{t - \tau_*(t,z)}{z}. \tag{7}$$

It is also easy to imagine the inverse problem solution in parametric form. It follows from (5) - (7) that if we know the field $v(t,z)$ at time t, at distance z from the input, then the input velocity field $v_0(\tau)$ is defined by the relations

$$v_0(\tau) = v(t,z), \qquad \tau = t + zv(t,z) \tag{8}$$

Now let us discuss the fundamental possibility of solving the inverse problem taking into account the formation of discontinuities. At vanishingly small viscosity ($\Gamma \to 0$), the solution of the Burgers equation still has the form (7), but $\tau_*(t,z)$ is now - the coordinate of the absolute maximum of the function $G(\tau,t,z)$ to the variable τ

$$G(\tau,t,z) = F_0(\tau) - \frac{(\tau - t)^2}{2z}, \qquad F_0(t) = \int^t V(t')dt' \tag{9}$$

By substituting the coordinate of the absolute maximum $\tau_*(t,z)$ in expression (7), we obtain the field $v(t,z)$ at time t at a distance z. For small z function $G(\tau,t,z)$ has a unique maximum for all t and, therefore, the field $v(t,z)$ is continuous. This solution corresponds to the solution of a simple wave and it is possible an unambiguous recovery of the input profile is possible. With increasing distance from the source (z height) for some t_m function $G(\tau,t,z)$ has an equal maxima at two points $[\tau_m, \tau_{m+1}]$ simultaneously. This time point t_m corresponds to the position of discontinuity. The entire section of the profile of the input range $[\tau_m, \tau_{m+1}]$ is absorbed by the shock.

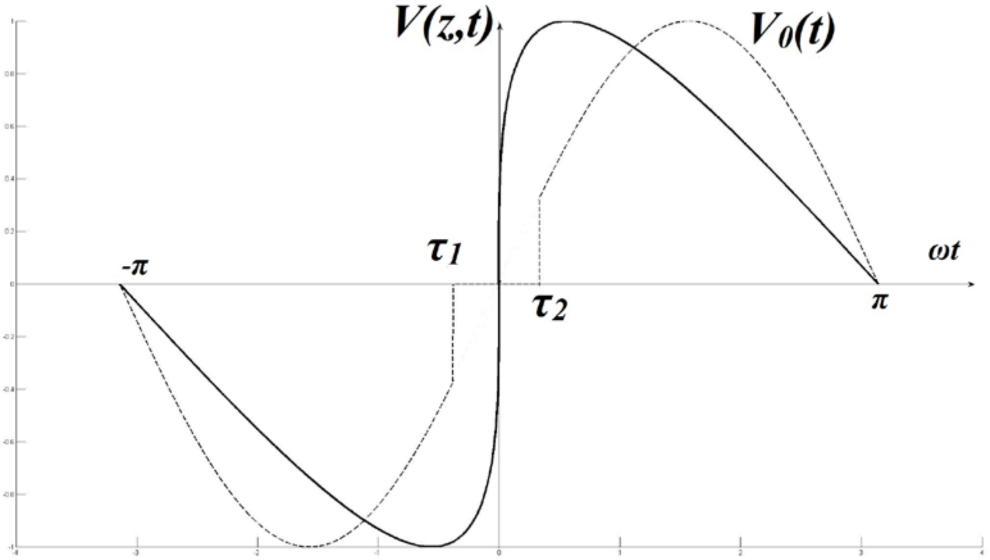

Figure 1. Restoring the initial profile with the account of shock front formation (formulas 6-8)

Thus, after the formation of shocks is fundamentally impossible to solve the inverse problem on the entire spatial range. These shock parameters (amplitude and velocity) give information only on the integral characteristics of the initial field in the interval $[\tau_m, \tau_{m+1}]$ [13]. The solution of the inverse problem is fundamentally different to regular and noise signals. For a regular signal, one can determine the region before the shock in which the evolution of the field is reduced to the simple wave equation, and the inverse problem has a unique solution. For Gaussian random fields in the input statistics shocks start at arbitrarily small distances from the entrance. In this case the exact solution of the inverse problem is fundamentally impossible. However, in [14] it was shown that at the initial stage of this effect is quite small, which gives hope to solve the inverse problem for a significant proportion of the time intervals of the initial perturbation.

3. The inverse problem of nonlinear acoustics in the frequency-distance domain.

In the time-distance domain we have the solution of the Riemann equation (4) in parametric (5),(6) or implicit form $v = v_0(t + zv)$. The spectral representation of the Burgers equation can be written as

$$\frac{\partial c(\omega, z)}{\partial z} - i\omega \frac{1}{2} F((F^{-1}(c(\omega, z)))^2) = -\Gamma \omega^2 c(\omega, z) \quad , \tag{10}$$

In equation (10), direct and inverse Fourier transform is used

$$c(\omega, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(t, z) e^{-i\omega t} dt = F(u), \qquad v(t, z) = \int_{-\infty}^{\infty} c(\omega, z) e^{i\omega t} dt = F^{-1}(c), \tag{11}$$

The spectral form of the Burgers equation (10) describes the evolution of the Fourier component $c(\omega, z)$ of intensive acoustic waves and is convenient for numerical modelling of regular and intensive noise acoustic signals. This equation can also be easily modified to describe the cylindrical and spherical waves [14].

At the initial stage of propagation and at zero viscosity ($\Gamma = 0$) one can write the explicit expression for the Fourier component $c(\omega, z)$ of Riemann wave. Using the parametric solution (5) we can rewrite the expression for the Fourier component $c(\omega, z)$ in the explicit form

$$c(\omega, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(t, z) e^{-i\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_0(\tau) e^{-i\omega T(\tau, z)} \frac{dT}{d\tau} d\tau, \qquad T(\tau, z) = \tau - z v_0(\tau) \tag{12}$$

and finally by the integration by the parts we get [15]

$$c(\omega, z) = -\frac{1}{2\pi i \omega z} \int_{-\infty}^{\infty} \left\{ \exp[-i\omega z v_0(t)] - 1 \right\} \exp(-i\omega t) dt \tag{13}$$

Thus, we have an explicit relation for the Fourier component $c(\omega, z)$ of the Riemann wave, but to find the initial velocity $v_0(t)$, or initial Fourier component $c_0(\omega)$, one needs to solve complex nonlinear equation (13).

Consider an initial task of restoring the power spectrum. For stationary random processes, spectral density $S(\omega)$ and the bispectral density $S_2(\omega, \omega_1)$ associated with random delta-correlated Fourier components of relationships is:

$$\begin{aligned} \left\langle c(\omega) c^*(\omega_1) \right\rangle &= S(\omega) \delta(\omega - \omega_1), \\ \left\langle c(\omega) c^*(\omega_1) c^*(\omega_2) \right\rangle &= S_2(\omega, \omega_1) \delta(\omega - \omega_1 - \omega_2) \end{aligned} \tag{14}$$

For Gaussian noise, the spectra of all the higher orders, in particular the bispectrum are equal to zero. As the wave propagates in a medium with a quadratic nonlinearity, new spectral components arise at the new harmonic frequencies $\omega = \omega_1 + \omega_2$. Thus, there is a denormalization of the input Gaussian process and its higher spectrum is not equal to zero. Moreover, from (14) we see that the bispectrum reflects the statistical correlation triple spectral components emerging from the nonlinear interaction of the harmonics. The role of bispectrum as an indicator describing the nonlinear effects becomes obvious if we write down the equation for the spectral density corresponding to the spectral form of the Karman-Howarth equations [16]:

$$\frac{\partial S(\omega,z)}{\partial z} = T(\omega,z) - \Gamma\omega^2 S(\omega,z), \qquad T(\omega,z) = \frac{\iota\omega}{2}(G_{2,1}(\omega,z) - G_{1,2}(\omega,z)) \qquad (15)$$
$$G_{2,1}(\omega,z) = \int_{-\infty}^{\infty} S_2(\omega,\Omega)d\Omega \qquad G_{1,2}(\omega,z) = \int_{-\infty}^{\infty} S_2(\omega-\Omega,\Omega)d\Omega$$

Thus, we see that the bispectra define the process of nonlinear energy transfer along the spectrum.

On the basis of (13) and (14) we can obtain expressions for the spectra and bispectra of random simple waves. In particular, for the spectrum of a simple wave with Gaussian initial conditions, we obtain the expression [3]

$$S(\omega,z) = \frac{e^{-\sigma^2\omega^2 z^2}}{2\pi\omega^2 z^2}\int_{-\infty}^{\infty}\left(e^{B(\tau)\omega^2 z^2} - 1\right)e^{-i\omega\tau}d\tau \qquad (16)$$

In expression (16) $B(\tau)$ is the correlation function of the velocity field at the input and $\sigma^2 = B(0)$, the variance of the input process. This expression describes the spectrum of a simple wave, for which there is no damping. However, the integration over all frequencies shows that (16) describes the decrease of the energy of a simple wave [3]. In [17-19] it was shown that this effect is due to the fact that the spectral representation of the solution of the Riemann equation (12) corresponds to the transition from a multi-valued solutions to the unambiguous one. The inverse Fourier transform of (12) is a single-valued solution which is the summation of the branches of multi-valued solutions with different characters and accurately simulates the damping at discontinuities. In [19] it was shown that at the initial stage this effect is quite small, which gives hope for the solution of the inverse problem.

Generally, for finding the initial spectrum $S_0(\omega)$ (correlation function $B_0(\tau)$) it is necessary to solve a nonlinear integral equation (16). Nevertheless, at the initial stage in the frequency range ($\sigma\omega z << 1$), the exponent in (16) and also in the expression for the bispecrum can be expanded in a series [3,9]

$$S(\omega,z) = S_0(\omega) + \frac{1}{2}\omega^2 z^2\left(S_0(\omega)\otimes S_0(\omega) - 2\sigma^2 S_0(\omega)\right)$$
$$T(\omega,z) = \omega^2 z\ \left(S_0(\omega)\otimes S_0(\omega) - 2\sigma^2 S_0(\omega)\right) \qquad (17)$$

Here $\otimes$ is the convolution operator. From (17) it follows that the independent measurements of the spectrum and bispectrum in section *z* allow us to restore the initial spectrum

$$S_0(\omega) = S(\omega,z) - \frac{1}{2}zT(\omega,z) \qquad (18)$$

For the numerical simulation of random evolution of intense acoustic waves, we used the Burgers equation in spectral form (10). To realize the numerical simulation based on equation (10), an iterative scheme was proposed using a fast Fourier transform, which enables the calculation of the wave profile, spectrum and bispectrum of nonlinear waves.

To simulate the noise *v (z, t),* a random number generator was used, which allowed us to form a Gaussian random process with zero mean and unit variance. The original process was set to the time step Δt = 0,001 sec, and the length of realization 2048 points (i.e., 2^{11}). The software for Fast Fourier Transform (direct and inverse) was taken from the MATLab library where numerical simulation was carried out. To smooth the frequency spectrum on the borders, the Hamming window was used, which allowed us to reduce the side lobes of the spectrum after filtering. The dimensionless Goldberg number was set to 0.01, which corresponded to the values of the Reynolds number equal to 100. On the basis of obtained implementations of Fourier images of the velocity field, standard MATLab programs were used to calculate the correlation functions and bispectra at different distances from the entrance. Averaging for spectral and bispectral characteristics was performed over 1,000 implementations. Further, based on expressions (15), the function *T (ω, z)* describing the nonlinear redistribution of energy over the spectrum was evaluated, and the input spectrum was restored using formula (18).

Figs. 2a and 2b present function *T (ω, z)* and initial spectrum $S_0(\omega)$ (an initial spectrum has maximum at the zero frequency), the spectrum at distance z from the entrance to the non-linear medium *S(ω,z)* and the restored the initial spectrum $\tilde{S}_0(\omega)$.

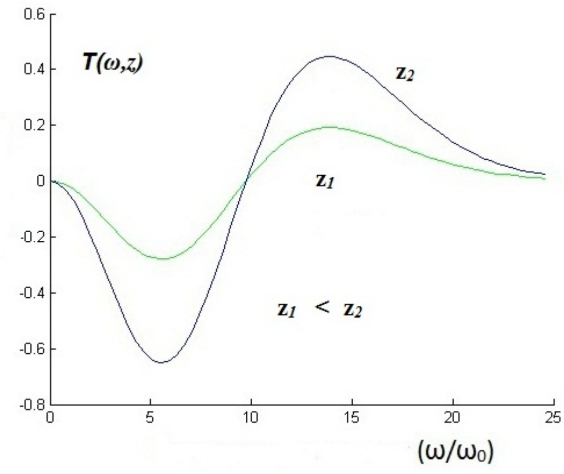

Figure 2a. Evolution of function *T (ω, z)* at the different distance z from the entrance to the non-linear medium ($z_2 > z_1$)

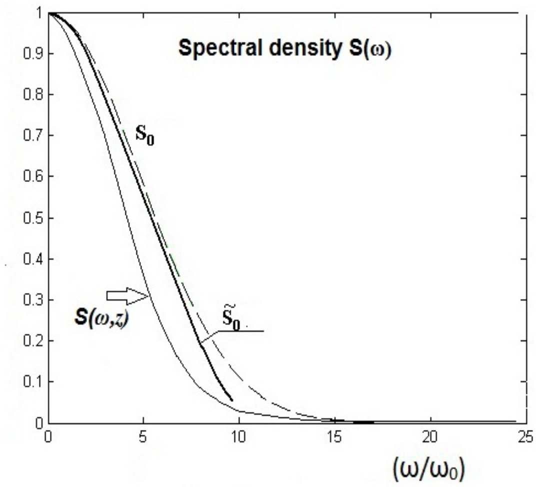

Figure 2b. Initial spectrum $S_0(\omega)$ (the spectrum has its maximum at the zero frequency), the spectrum at the distance z from the entrance to the non-linear medium *S(ω,z)* and the restored the initial spectrum $\tilde{S}_0(\omega)$.

Numerical simulations have shown quite good restoration of the initial spectrum at low frequencies.

CONCLUCIONS

The inverse problem in the distance-time domain and in the distance-frequency domain is considered. It is shown that the complete solution of the inverse problem in nonlinear acoustics is fundamentally impossible because of the discontinuities. However, at the initial stage of the intense acoustic noise propagation, the use of bispectra allows a good recovery of the initial spectrum in a sufficiently wide frequency range.

ACKNOWLEDGMENTS

This work was supported by a grant from the Russian Science Foundation No. 14-12-00882.

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