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HIGH ORDER ANSATZ FUNCTIONS ON ISOGEOMETRIC BOUNDARY ELEMENTS

Sören Keuchel, Nils Christian Hagelstein, Olgierd Zaleski, Otto von Estorff
Novicos GmbH, 21073 Hamburg, Germany
email: keuchel@novicos.de , hagelstein@novicos.de , zaleski@novicos.de , estorff@novicos.de

In an Isogeometric Analysis (IGA), exact CAD geometries are used directly as a basis for numerical simulations. This procedure can be incorporated to the Boundary Element Method (BEM) to achieve a very high accuracy. Since the error due to discretization of the geometry is eliminated, other influences tend to have a higher impact on the overall solution quality. Besides an appropriate numerical integration scheme, one of the most significant steps is the approximation of the boundary values. In ordinary BEM formulations constant, linear or at most quadratic Lagrange polynomials are used to represent the values on the surface. The employed NURBS based geometry description allows for a variety of different ansatz functions for the boundary values. In general, the shape of the functions and the supporting points can be arbitrarily set on the exactly defined surfaces. This enables the use of high order ansatz functions, which leads to more favorable error convergence rates in contrast to a refinement of elements with the same order. The High Order IGABEM is applied to acoustic problems in the frequency domain. In the contribution the governing relations are introduced and the numerical examples show the achieved efficiency.

Boundary Element Method, Isogeometric Analysis, Acoustics, High Order

1. Introduction

An Isogeometric Analysis (IGA) eliminates the approximation of the geometry and computes the numerical solution directly on the CAD description, which is based only on the surface [1]. Therefore, the combination with the Boundary Element Method (BEM), which also relies only on the surface, is advantageous. This conjunction shall be used to allow the design engineer to have an early judgment, whether changes of the product have a positive effect or not. This is only possible if the numerical simulation is directly incorporated to the development stage without a need of a demanding mesh preparation, as in the conventional procedure.

The field of acoustical problems is a good example, where this new IGABEM formulation can be applied and a detailed explanation is given in [5]. The IGA concept imposes new challenges in contrast to previous BEM formulations. No approximation of the geometry is incorporated and therefore other BEM steps become more important. On the one hand, the numerical integration becomes more significant, which is described in [4]. On the other hand, the approximation of the boundary values become more crucial. These values are described by some ansatz function, e.g. Lagrange or Bernstein polynomials of arbitrary order, and corresponding supporting points. Additionally, the values of these points depend on the placement of the collocation points. The approximation of the boundary values with high order ansatz function is the topic of the current contribution, while the formulation from [4] is used as a basis. The following sections describe the governing relations and numerical examples show the behavior of the formulation. In the last section the results are concluded.

2. Boundary Element Method in Acoustics

"In the frequency domain, the Helmholtz equation

$$\nabla^2 p = -k^2 p \tag{1}$$

is the governing partial differential equation that describes the propagation of waves. The equation consists of the sound pressure p and the wavenumber $k = \frac{\omega}{c}$ with the speed of sound c. A transformation onto the surface Γ leads to the conventional boundary integral equation (CBIE)

$$c(\mathbf{x})p(\mathbf{x}) = \int_\Gamma \left[G(\mathbf{x},\mathbf{y})q(\mathbf{y}) - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n_y}} p(\mathbf{y}) \right] \mathrm{d}\Gamma_y \; , \tag{2}$$

where $\mathbf{x}$ is the receiver point, $\mathbf{y}$ is the source point, $\mathbf{n_y}$ is the normal vector at $\mathbf{y}$, and $G(\mathbf{x},\mathbf{y})$ is the fundamental solution. For the three dimensional case this fundamental solution reads

$$G(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \; . \tag{3}$$

The boundary values $p(\mathbf{y})$ and $q(\mathbf{y})$ are discretized by a supporting point f_m with its assigned ansatz-function $\phi_m(\xi)$

$$f(\xi) = \sum_{m=1}^{M} \phi_m(\xi) f_m \; . \tag{4}$$

The definition of the ansatzfunction leads to the used concept, either the classical Lagrange concept or the new IGA concept. A drawback of the CBIE is its non-uniqueness problem that corresponds to the occurrence of the so called spurious eigenfrequencies. The problem can be healed by the Burton-Miller formulation [6], which is based on the linear combination of the CBIE and the hypersingular boundary integral equation (HBIE)

$$c\left(\mathbf{x}\right) q\left(\mathbf{x}\right) = \int_\Gamma \left[\frac{\partial G\left(\mathbf{x},\mathbf{y}\right)}{\partial \mathbf{n_x}} q\left(\mathbf{y}\right) - \frac{\partial^2 G\left(\mathbf{x},\mathbf{y}\right)}{\partial \mathbf{n_x}\partial \mathbf{n_y}} p\left(\mathbf{y}\right) \right] \mathrm{d}\Gamma_y \; . \tag{5}$$

The difficulty of this equation is the strongly and hypersingular term, which require special integration routines. An interesting procedure to overcome this problem for the Isogeometric BEM is explained in a following section. The system of equation is created by a collocation method. An overview with a focus on iterative solvers for BEM systems is given in [2]." [4]

3. Isogeometric Analysis

"In an Isogeometric Analysis the description of the geometry shall be used as the representation of the field variables. In the case of a BEM, these are the boundary values. The common standard in the CAD industry are NURBS that allow a rather simple description of complex shapes. NURBS surfaces are defined as patches that can consist of different elements. In general, the ansatz functions are defined on these patches and can differ between the elements. Additionally, the functions are defined recursively, which increases the numerical complexity for the evaluation significantly. Hence, the Bezier extraction is applied to achieve the same shape functions on each element [5]. Within this procedure, supporting points are introduced that change the shape of the ansatz function in the parameter space, but the geometry in global coordinates is kept. The surfaces are defined as a multiplication of two one dimensional functions of the form

$$R_i^p(\xi) = \frac{B_i^p(\xi) w_i}{\sum_{j=1}^{n} B_j^p(\xi) w_j} \; . \tag{6}$$

The Bezier extraction allows the direct evaluation of the B-Splines as Bernstein polynomials

$$B_i^p(\xi) = \frac{1}{(b-a)^p} \binom{p}{i} (\xi - a)^i (b - \xi)^{p-i} \; . \tag{7}$$

The positioning of the collocation points has a large influence on the solution accuracy. For the presented ansatz function, the placement is independent of the shape functions, since the functions are not interpolatory at the collocation points. This is in contrast to a Lagrange formulation, where the ansatzfunction is 1 at the corresponding supporting point." [4]

In this contribution the shape of the functions and the placement of the collocation points shall be investigated. The NURBS are compared to the Lagrange polynomials. The placement of the collocation points is either equidistant or at the zeros of the Legendre polynomials, as depicted in figure 1 for a quadratic ansatz. A further remark has to be made on triangular isogeometric elements. These elements are usually not supported in CAD software and are formulated as a collapsed quadrangle. Therefore, when using a placement of the collocation points in parameter space, a high density of collocation points in one corners arises. To overcome this problem these corner points are unified for the NURBS ansatz function. Only one degree of freedom represents these points and the corresponding shape function is the addition of the previous functions of the points. For the Lagrange polynomials this procedure is not applied. Additionally, the high density of collocation points needs a special treatment in the numerical integration, which is explained in the next section.

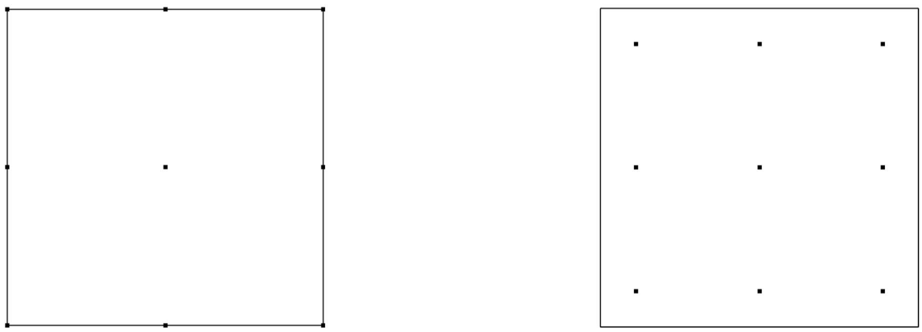

Figure 1: Collocation point placement: equidistant (left), zeros of Legendre polynomials (right)

4. Numerical Integration

"The IGA concept allows an exact representation of the geometry and therefore can increase the solution accuracy significantly. In previous formulations, it was satisfying to compute all the values in the same range as the discretization error of the geometry. Due to the elimination of this approximation, the other influences on the solution quality play a more considerably role. One of the influences is the numerical integration, which has an extraordinary impact on the solution quality in the BEM. The CBIE and HBIE include weakly singular, strongly singular and hypersingular terms that have to be treated carefully. If the collocation point and the source element are well separated, a conventional Gaussian integration is applicable. When they are on the same element, the integration becomes singular. A further important type is the quasi singular integration, where the collocation is very close to the source element, but is not lying on it. A difficulty of this type is that no shared local coordinate system exists. For the IGA, the curved surfaces increase this difficulty once more. The integration routines are based on [3] and the main changes are addressed in the following.

The singular integration is done in the sense of Guiggiani's formulation [7]. It removes the singularity by a subtraction of a series expansion from the kernel. This is possible, since the series expansion has the same order of singularity as the kernel and approximates the static fundamental solution. Afterwards, the subtracted term is added back and can be integrated semi-analytically. The procedure leads to a direct evaluation of all singularities for two collocation points lying on the same element. The changes of the ansatz function to NURBS have to be addressed in the required derivatives of the series expansion.

The treatment of the quasi-singular integrals is a sinh-transformation [8], that reduces the singular character. The different local coordinates systems of the two considered collocation points are the challenge of this procedure. The point on the source element with the shortest between the collocation point and source element is needed in local coordinates. For plane Lagrange elements the transformation onto the other local coordinate system, firstly, an orthogonal projection is used and, secondly, the non-linear transformation from local to global coordinates has to be inversed. The inversion is solved by a multi-dimensional Newton-Raphson method. Due to the curved element description by NURBS, the orthogonal projection cannot be applied. Therefore, the approximation of the Newton-Raphson method is already used to achieve the point of the shortest distance in global coordinates." [4]

5. Results

In this section the presented High Order IGABEM shall be investigated. The influence of the shape functions and the placement of the collocation points is analyzed on a sphere with 6 triangular and 6 quadrangular NURBS elements. Due to the occurrence of the triangles at the poles of the sphere, the previously mentioned problem of a high collocation point density is addressed. The interaction between the high order ansatz function and an accurate numerical integration is depicted as well.

The analyzed case is a sphere with a radius of $R = 0.5$ m and a monopole source at $\mathbf{x}_{\mathbf{mono}} = [-0.2\,/\,-0.2\,/\,-0.2]$ that is used to prescribe the Neumann boundary conditions and compare the numerical solution to an analytical pressure. The acoustical medium is air and the frequency is set to $f = 200$ Hz. The corresponding Dirichlet error

$$e_D = \frac{\| \left(\mathbf{p}_{num} - \mathbf{p}_{ref}\right) \|}{\|\mathbf{p}_{ref}\|} \tag{8}$$

with

$$\|\mathbf{p}\| = \sqrt{\int_\Gamma |\mathbf{p}|^2 \mathrm{d}\Gamma} \tag{9}$$

is integrated over the elements to take into account the higher order ansatz functions.

5.1 Comparison of the Ansatz Function and the Placement of the Collocation Points

In figure 2 the Dirichlet error over the degrees of freedom (dof) is plotted. An h-refined description with a constant boundary value approximation is used as as reference ("p0 h-refined"). The NURBS and Lagrange ansatz function are compared and also the equidistant and Legendre placement of the collocation points are investigated. All computations in this subsection are done without the sinh-transformation and with an integration order of $g = 10 \times 10$ for the near interaction and $g = 3 \times 3$ for the far interaction. The reference shows the expected behavior of the error reduction. For the Legendre placement the results for both function types are good up to the element order $p = 2$, where a steeper decrease than the reference can be achieved. Above this element order the errors increase again, which is in contrast to the expected behavior. The equidistant placement does not achieve good results. Therefore, the behavior Legendre placement is analyzed more precisely. In figure 4 the error distribution over the sphere for the element order $p = 3$ is depicted. The Lagrange ansatz functions have a bigger spot of high errors at the poles, which relies on the different placement strategy for a triangle. For the NURBS ansatz function there is only one dof that represents the collapsed edge. The figures approve that this strategy has some advantages, but the reason for the high overall error cannot be found. Hence, in the next section the influence of the numerical integration in combination with high order elements is focused.

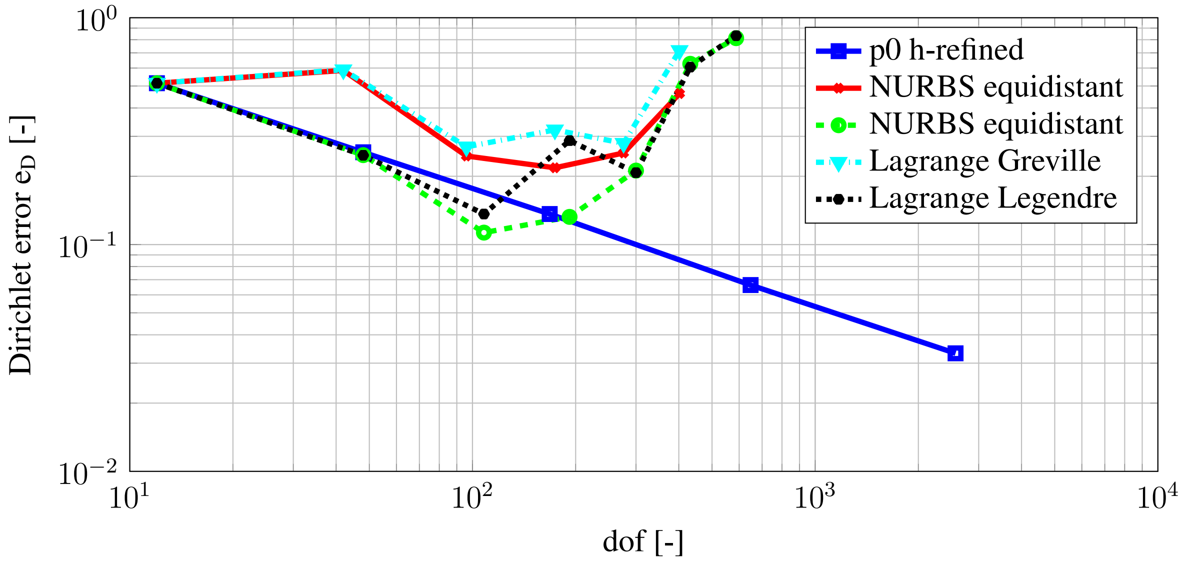

Figure 2: Dirichlet error over degrees of freedom (dof) of a sphere and an increasing element order from $p = 0$

5.2 Influence of the Numerical Integration for High Order Elements

In figure 3 the Dirichlet error over the degrees of freedom (dof) is plotted and shows the same reference as in the previous subsection. Additionally, two computations with the sinh-transformation and with an increased integration order of $g = 40 \times 40$ for the near interaction are introduced ("Int.") and compared to the ones with the lower integration order from the previous subsection.

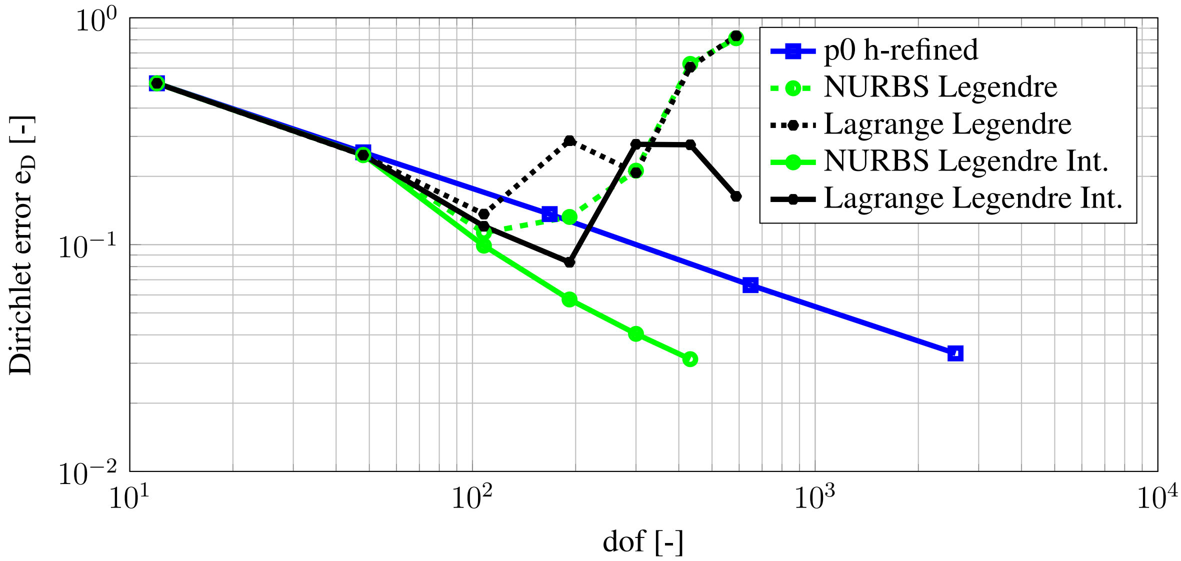

Figure 3: Dirichlet error over degrees of freedom (dof) with higher integration order of a sphere and an increasing element order from $p = 0$

It is clearly visible, that the accuracies for the orders above $p = 2$ are increased, but only the "NURBS Legendre Int." combination achieves the expected error decrease for all element orders. To find a better insight to this behavior, the distribution of the error for $p = 3$ and $p = 5$ with the higher integration order is shown for both ansatz function types in figure 5 and figure 6. In comparison to the previous subsection, the overall error is decreased significantly for $p = 3$, which emphasizes the need for special integration routines for high order isogeometric elements. With an increase of the order to $p = 5$ the Lagrange polynomials show again the problem at the poles, that correlates to a high collocation point density due to the collapsed edge of the isogeometric triangles. Hence, the high order triangular isogeometric elements require special treatment on top of the high numerical integration order. Using the achieved knowledge, the High Order IGABEM is capable of achieving the same error with less dof than the constant formulation. In this case, only roughly 400 dof are necessary to attain the same error as 2400 dof with the constant formulation. Although, the factor of 6 is not directly transferable to the time due to the larger effort of the numerical integration, it can directly be relayed to the amount of memory.

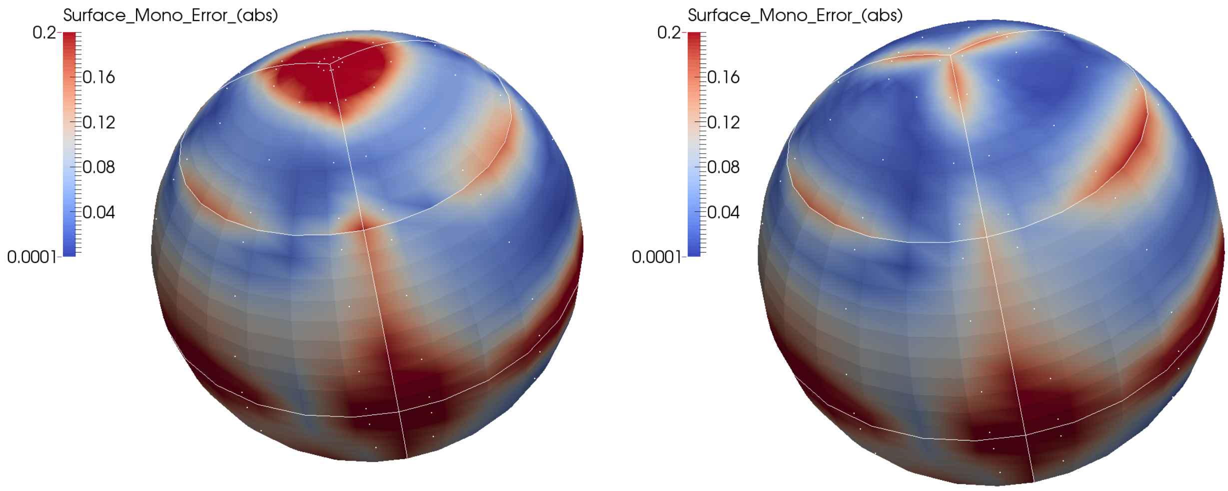

Figure 4: Error distribution over the sphere with $p = 3$ and a standard integration order for the Lagrange (left) and the NURBS (right) ansatz

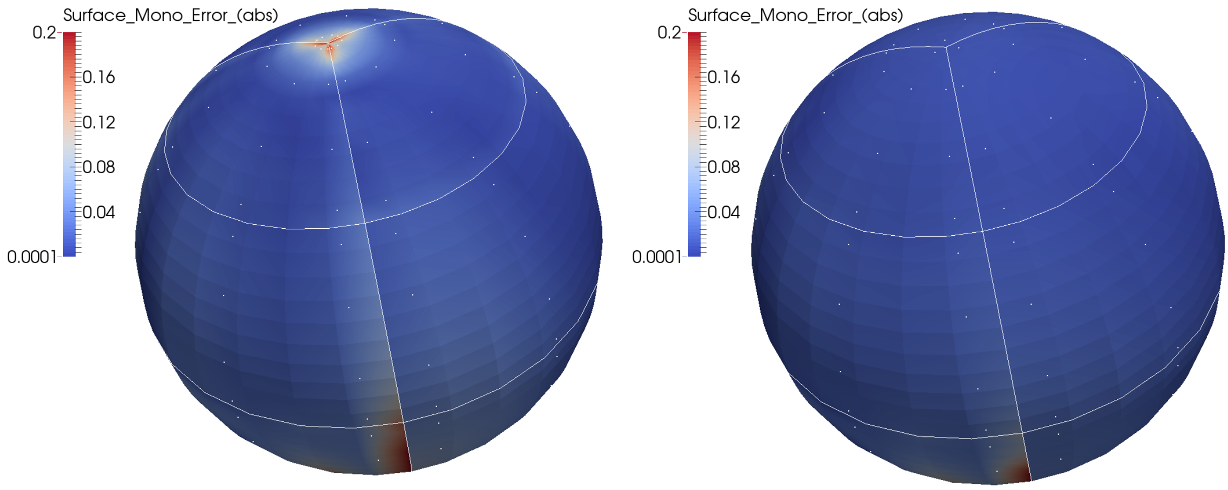

Figure 5: Error distribution over the sphere with $p = 3$ and a high integration order for the Lagrange (left) and the NURBS (right) ansatz

6. Conclusion

A High Order IGABEM was presented that enables the computation of acoustical problems with the same accuracy but fewer elements than the constant element formulation. The governing relations for the description of exact geometries were given and special integration routines were presented. The results show the behavior in terms of the shape of the ansatz functions and the placement of the

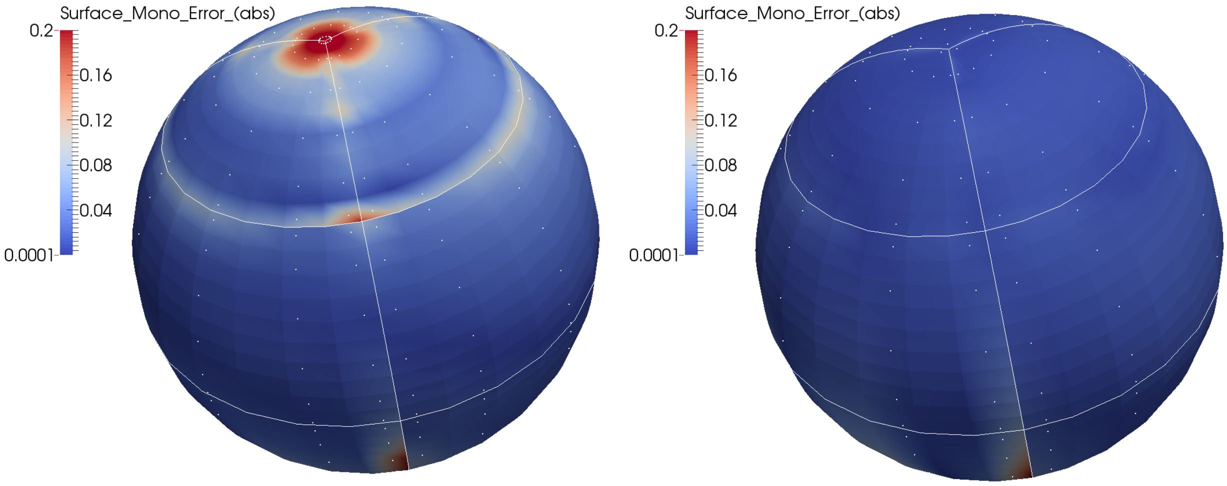

Figure 6: Error distribution over the sphere with $p = 5$ and a high integration order for the Lagrange (left) and the NURBS (right) ansatz

collocation points. The zeros of the Legendre polynomials show a better behavior than the equidistant placement for the Lagrange polynomials as well as for the NURBS ansatz functions. Furthermore, a significant increase of the accuracy can only be achieved for a high integration order in corporation with the sinh-transformation, whereas the NURBS shape functions are better than the Lagrange polynomials. This behavior is traced back to the collocation point placement at a collapsed edge of a triangular isogeometric element, where a unifying of the collocation point row at this edge shows an advantage. Taking the obtained knowledge into account, the big potential of the presented formulation is approved.

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