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INTERPLAY BETWEEN TIME SERIES ANALYSIS AND SPATIAL SERIES ANALYSIS

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ABSTRACT

In this paper, we present a review of the basic notions of time series analysis and spatial series analysis, as well as the interplay between them. It is demonstrated that wavenumber-frequency processing is a natural generalization of these two analyses. The relation between the single-channel and multichannel cases is briefly discussed.

We start with the Fourier relations for infinite duration, continuous-time series and the corresponding spatial series. In practical applications, we are limited in both available data and processing ability; hence, we have to estimate the desired quantities from a finite data record. Also, when using a digital computer for our calculations, we are forced to employ sampled data. A brief discussion of time-space series modelling and analysis is included.

I. INTRODUCTION

The analogy of spatial waves with temporal signals makes it possible to treat them both in a similar manner. A time series is a set of observations generated sequentially in time. In particular, a time series may be thought of as one particular realization of a stochastic process [1]. For a complex-valued zero-mean, wide-sense stationary and ergodic process represented by the time series h(t), $-\infty < t < \infty$, the autocorrelation function is defined by [2]

$$R_h(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h^*(t)h(t+\tau)dt \quad (1)$$

where the asterisk signifies complex conjugation.

Suppose that the time series h(t) is passed through an ideal narrow-band filter with bandwidth Δf, centered at f_c, and with unity gain. Let $h(t,f_c,\Delta f)$ denote the resulting filter output. The mean square value of the filter output is given by

$$\Psi_h^2(f_c,\Delta f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h^2(t,f_c,\Delta f)dt \quad (2)$$

The power spectral density of the original process is defined by [2]

$$S_h(f) = \lim_{\Delta f\to 0} \frac{\Psi_h^2(f,\Delta f)}{\Delta f}$$

$$= \lim_{\Delta f\to 0} \frac{1}{\Delta f}[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h^2(t,f,\Delta f)dt] \quad (3)$$

The autocorrelation function $R_h(\tau)$ and the power spectral density function $S_h(f)$ form a Fourier transform pair, as shown by the pair of equations

$$S_h(f) = \int_{-\infty}^{\infty} R_h(\tau)e^{-j2\pi f\tau}d\tau \quad (4)$$

$$R_h(\tau) = \int_{-\infty}^{\infty} S_h(f)e^{j2\pi f\tau}df \quad (5)$$

Equations (4) and (5) are basic relations in the theory of spectral analysis of random processes, and together they constitute the Wiener-Khintchine theorem [2].

Let us now consider a travelling plane wave propagating in space with speed v in a certain direction. At some particular instant of time, t_o, the magnitude of the wave is a function of its position in space. Let Q be a spatial point determined by the position vector, $\vec{z}$, with respect to some arbitary origin, as shown in Fig. 1. The magnitude of the wave at point Q is denoted by $g(\vec{z}) = g(x_Q, y_Q, z_Q)$, indicating dependence on all three spatial coordinates.

For a spatially stationary (isotropic) signal field $g(\vec{z})$, we define the space correlation function at time t_o as

$$R_g(\vec{r}) = \lim_{\substack{L_x \to \infty \\ L_y \to \infty \\ L_z \to \infty}} \frac{1}{8L_x L_y L_z} \int_V g^*(\vec{z}) g(\vec{z}+\vec{r}) d\vec{z} \qquad (6)$$

where the triple integral is over the volume V of a cube with the edges $2L_x$, $2L_y$ and $2L_z$, and $\vec{r}$ is the spatial lag vector. The wavenumber spectrum of the wave $g(\vec{z})$ is defined in a manner similar to the power spectral density of the temporal time series.

Suppose that the multitude of plane waves is impinging on a vertical line array from different elevation angles θ, where θ is measured with respect to the normal to the array. If the array sensors are sufficiently closely spaced, we can consider the array as a narrow-beam spatial filter in coordinate z, with beamwidth Δθ, centered at θ_c, and with unity gain. Let $g(d, \sin\theta_c, \sin\Delta\theta)$ denote the resulting filter output, where d is the distance along the z-axis from the origin. The mean square value of the filter output $\Psi_g^2(\sin\theta_c, \sin\Delta\theta)$ can be defined in the same manner as the mean square value of the temporal filter output, given by Eq. (2). In general, the mean square value and the corresponding wavenumber power spectral density will depend on all three spatial coordinates, or, equivalently, on the wavenumber vector $\vec{\nu}$. Thus, by analogy with Eq. (3), we may write

$$S_g(\vec{\nu}) = \lim_{\Delta\vec{\nu} \to 0} \frac{1}{\Delta\vec{\nu}} \cdot \left[\lim_{V \to \infty} \frac{1}{V} \int_V g^2(\vec{z}, \vec{\nu}, \Delta\vec{\nu}) d\vec{z}\right] \qquad (7)$$

where V is a cube in space centered at the origin. The vector wave-number $\vec{\nu}$ plays a role similar to the scalar frequency variable, f. Thus, the Wiener-Khintchine theorem in spatial domain can be written as

$$S_g(\vec{\nu}) = \int_{V_r} R_g(\vec{r}) e^{-j2\pi(\vec{\nu}\cdot\vec{r})} d\vec{r} \qquad (8)$$

$$R_g(\vec{r}) = \int_{V_\nu} S_g(\vec{\nu}) e^{j2\pi(\vec{\nu}\cdot\vec{r})} d\vec{\nu} \qquad (9)$$

where the dots in the exponents indicate vector dot products. The ranges of integration in Eqs. (8) and (9) are infinite-extent three-dimensional spaces, spanned by the spatial lag vectors $\vec{r}$ and the wavenumber vectors $\vec{\nu}$, respectively.

Presenting travelling waves as being dependent on spatial position only, and not being dependent on time, is only of academic interest; we used it above merely to stress the direct analogy between spatial sereis and time series. In practice, travelling waves are also time dependent, and so we write the time-space series as $s(t, \vec{z})$ or equivalently, $s(t,x,y,z)$.

For a time-space series which is wide-sense stationary in both time

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and space (homogeneous), we define the time-space correlation function and the frequency-wavenumber spectrum as follows, respectively,

$$R_s(\tau,\vec{r}) = \lim_{\substack{T\to\infty \\ L_x\to\infty \\ L_y\to\infty \\ L_z\to\infty}} \frac{1}{16TL_xL_yL_z} \cdot \int_{-T}^{T}\int_{V} s^*(t,\vec{z})s(t+\tau,\vec{z}+\vec{r})d\vec{z}\, d\vec{r} \qquad (10)$$

and

$$S_s(f,\vec{\nu}) = \lim_{\substack{\Delta f\to 0 \\ \Delta\vec{\nu}\to 0}} \frac{1}{\Delta f \Delta\vec{\nu}} [\lim_{\substack{T\to\infty \\ V\to\infty}} \frac{1}{2TV} \cdot \int_{-T}^{T}\int_{V} s^2(t,f,\Delta f,\vec{z},\vec{\nu},\Delta\vec{\nu})d\vec{z} \qquad (11)$$

where all the quantities have the same meaning as in Eqs. (1), (3), (6) and (7). Similarly, the Wiener-Khintchine relations between the time-space correlation function $R_s(\tau,\vec{r})$ and the frequency-wavenumber spectral density $S_s(f,\vec{\nu})$ can be written as a four-dimensional Fourier transform pair [4]

$$S_s(f,\vec{\nu}) = \int_{-\infty}^{\infty}\int_{V_r} R_s(\tau,\vec{r})e^{-j2\pi(f\tau+\vec{\nu}\cdot\vec{r})}d\vec{r}\, d\vec{\tau} \qquad (12)$$

$$R_s(\tau,\vec{r}) = \int_{-\infty}^{\infty}\int_{V_\nu} S_s(f,\vec{\nu})e^{j2\pi(f\tau+\vec{\nu}\cdot\vec{r})}d\vec{\nu}\, d\vec{f} \qquad (13)$$

We also define the cross-spectrum $P_x(f,\vec{r})$ as the Fourier transform of $R_s(\tau,\vec{r})$ with respect to τ, that is,

$$P_x(f,\vec{r}) = \int_{-\infty}^{\infty} R_s(\tau,\vec{r})e^{-j2\pi f\tau}d\tau \qquad (14)$$

A wave component of frequency f and vector wavenumber $\vec{\nu}$ has a vector propagation constant $\vec{k} = 2\pi\vec{\nu}$, wavelength $\lambda = 1/|\vec{\nu}|$, and velocity of propagation $\vec{v}$ with the direction of $-\vec{\nu}$ and the magnitude $|\vec{v}| = f/|\vec{\nu}| = f\lambda$.

Finally, we note that the time-space series $s(t,\vec{z})$ belongs to a class of so-called "multivariate" signals, i.e., signals which depend on more than one independent variable. In our case, there are one temporal and three spatial variables, a total of four variables. On the other hand, $s(t,\vec{z})$ is often treated as a set of K time series at K discrete array sensors positioned in space, that is we have $s_k(t) = s(t,\vec{z}_k)$, $k = 1, 2, \ldots, K$. In this case, we are dealing with so-called "multichannel" analysis, i.e., simultaneous analysis of K univariate signals which depend on a temporal variable only.

Table 1 Equivalence of Temporal and Spatial Analysis.

TEMPORAL	SPATIAL
time, t	spatial position vector, $\vec{z}$
temporal lag, τ	spatial lag vector, $\vec{r}$
frequency, f	wavenumber vector, $\vec{\nu}$
autocorrelation $R(\tau)$	cross-spectrum at f_0, $P(f_0,\vec{r})$
power spectrum $S(f)$	wavenumber spectrum at f_0, $S(f_0,\vec{\nu})$

Table 1 summarizes an equivalence between the basic quanitities in the temporal analysis and the spatial analysis.

II. LINEAR FREQUENCY-WAVENUMBER ANALYSIS OF TIME-SPACE SERIES

We now present the basics of the linear methods of spectral analysis of a time-space series that is limited in both time and spatial coordinates. Our presentation closely follows the excellent exposition given by McDonough [4].

Array of sensors used in the reception of travelling waves represents the natural spatial sampling mechanism. After the signals at all sensors are frequency-translated into the desired domain (IF of baseband), a temporal processing can be performed in a usual way.

Assume K array sensor arbitrarily positioned in space, positions of which are determined by vectors $\vec{z}_k$, k = 1, 2, ..., K. We now have K data signals $x(t,\vec{z}_k)$, corresponding to each sensor, so that we are dealing with multichannel spectral analysis. Thus, for any pair (k,l) of sensor signals, we can specify the discrete-time estimate of the cross-correlation function

$$\hat{C}_x(m,\vec{z}_k,\vec{z}_l) = \frac{1}{N}\sum_{n=1}^{N-m} x^*(n,\vec{z}_k)x(n+m,\vec{z}_l) \quad (15)$$

for m = 0, 1, ..., M and k,l = 1, 2, ..., K, where N is the total number of temporal samples used in the analysis and M is the maximum temporal lag.

The Fourier transform of $\hat{C}_x(m,\vec{z}_k,\vec{z}_l)$ in time domain is the estimate $\hat{P}_x(f,\vec{z}_k,\vec{z}_l)$ of the cross-spectrum, Eq. (14). The indirect, Blackman-Tukey estimate [3], $P(f,\vec{z}_k,\vec{z}_l)$, is obtained by weighting $\hat{C}(f,\vec{z}_k,\vec{z}_l)$ of Eq. (15) and taking the finite Fourier transform, yielding

$$\hat{P}_x(f,\vec{z}_k,\vec{z}_l) = \sum_{m=-M}^{M} \hat{C}_x(m,\vec{z}_k,\vec{z}_l)w(m)e^{-j2\pi fm} \quad (16)$$

In order to apply the direct or periodogram method [5,6], we first compute the row "cross-periodogram" in frequency-space domain, as shown by

$$J_x(f,\vec{z}_k,\vec{z}_l) = X_k^*(f)X_l(f)$$

$$= \frac{1}{N^2}\left[\sum_{n=1}^{N} x_k(n,\vec{z}_k)e^{-j2\pi fn}\right]\left[\sum_{n=1}^{N} x_l(n,\vec{z}_l)e^{-j2\pi fn}\right] \quad (17)$$

and then smooth it with an appropriate window function W(f), as shown by

$$\hat{P}_x(f,\vec{z}_k,\vec{z}_l) = \int_{-\infty}^{\infty} J_x(\theta,\vec{z}_k,\vec{z}_l)W(f-\theta)d\theta \quad (18)$$

Cross-spectrum estimators of Eqs. (16) and (18) are identical. Direct approach is more frequently used since it is computationally more efficient. In this section, we assume that there is sufficient number of data samples N, so that segmenting the data and averaging the periodograms can be performed to provide variance reduction.

We now proceed to find an estimate of the frequency-wavenumber power spectral density, given by Eq. (10), for a homogeneous travelling wave. We shall make use of the estimate of Eq. (18) of the "frequency-space" cross-spectrum. Only the direct method will be considered. The procedure involves the following steps: (1) Compute the Fourier transform of the signals

$x(n,\vec{z}_k)$, $k = 1, 2, \ldots, K$, as follows:

$$X(f,\vec{z}_k) = \frac{1}{N} \sum_{n=1}^{N} x(n,\vec{z}_k) e^{-j2\pi fn} \quad (19)$$

(2) Next, compute the periodogram in the frequency-wavenumber domain by

$$I_x(f,\vec{\nu}) = \frac{1}{K} \left| \sum_{k=1}^{K} X(f,\vec{z}_k) e^{-j2\pi \vec{\nu}\cdot\vec{z}_k} \right|^2 \quad (20)$$

where K is the number of array sensors.

(3) Smooth the periodogram (20) by a four-dimensional window in frequency and wavenumber $W(f,\vec{\nu})$, as shown by

$$\hat{S}_x(f,\vec{\nu}) = \int_{-\infty}^{\infty} \int_{V_\nu} W(f-\theta,\vec{\nu}-\vec{\eta}) I_x(\theta,\vec{\eta}) d\vec{\eta} d\theta \quad (21)$$

where V_ν is an infinite volume in $\vec{\nu}$-space.

We can simplify expressions for $I_x(f,\vec{\nu})$ and $S_x(f,\vec{\nu})$ by assuming that the time-space window $w(n,\vec{z})$ can be factored as

$$w(n,\vec{z}) = w_h(n) w_r(\vec{r}) \quad (22)$$

Then, the four-dimensional Fourier transform also factors as

$$W(f,\vec{\nu}) = W_f(f) W_\nu(\vec{\nu}) \quad (23)$$

The estimate $\hat{S}_x(f,\vec{\nu})$ can now be written as

$$\hat{S}_x(f,\vec{\nu}) = \int W_\nu(\vec{\nu}-\vec{\eta}) Q_x(f,\vec{\eta}) d\vec{\eta} \quad (24)$$

where

$$Q_x(f,\vec{\nu}) = \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{K} \hat{P}_x(f,\vec{z}_k,\vec{z}_l) \cdot e^{-j2\pi\vec{\nu}\cdot(z_k - z_l)} \quad (25)$$

and $\hat{P}_x(f,\vec{z}_k,\vec{z}_l)$ is given by Eq. (18).

Often, the discrete-space version of the estimator of Eq. (24) is simply written in the following form [7]

$$\hat{S}_x(f,\vec{\nu}) = \sum_{k=1}^{K} \sum_{l=1}^{K} \hat{P}_x(f,\vec{z}_k,\vec{z}_l) \gamma_k^*(\vec{\nu}) \gamma_l(\vec{\nu}) \quad (26)$$

where

$$\gamma_k(\vec{\nu}) = w_k e^{j2\pi\vec{\nu}\cdot\vec{z}_k}, \quad k = 1, 2, \ldots, K \quad (27)$$

with w_k being available for ajustment in some way to improve for estimator properties. Often, $w_k = 1$ is used and the estimator of Eq. (26) is referred to as delay-and-sum beamformer [7].

A number of temporal samples are usually taken to be large enough, so that both good resolution and small variance of the $P_x(f,\vec{z}_k,\vec{z}_l)$ are achieved. To obtain comparable performance in wavenumber domain, spatial extension of apparatus' dimensions is required. The alternative is to use nonlinear spectral estimator algorithms which we discuss next.

III. NONLINEAR FREQUENCY-WAVENUMBER ANALYSIS AND SIGNAL MODELLING

In this section, we briefly mention some new nonlinear methods of frequency-wavenumber spectral analysis.

The maximum likelihood (ML) method, introduced by Capon [8], can be formulated in the following way. The ML filter is that filter which passes signal at frequency component f_n undistorted while suppressing all other components, including noise. The output power of that filter is minimized under the constraint that the signal at frequency f_n is passed undistorted. The obtained minimum power is the actual ML estimate of the frequency-wavenumber spectrum and is given by [4,8]

$$\hat{S}_{x(ML)}(n,\vec{v}) = [\underline{e}^{H}(\vec{v})\hat{\underline{P}}_{x}(n)^{-1}\underline{e}(\vec{v})]^{-1} \quad (28)$$

where $\underline{e}(\vec{v})$ is a "beam-steering" column matrix at frequency, f_n, given by

$$\underline{e}^{T}(\vec{v}) = [e^{j2\pi\vec{v}\cdot\vec{z}_1}, \ldots, e^{j2\pi\vec{v}\cdot\vec{z}_K}], \quad (29)$$

and $\hat{\underline{P}}_x(n)$ is an estimate of the cross-spectral matrix whose l,k-th element is given by Eqs. (16) or (18), and K is the number of array sensors.

It was demonstrated that the formulations of the estimator of Eq. (25) as a maximum likelihood estimator and as a minimum variance unbiased estimator, both lead to the same result [4,8]. The resolution properties surpass those of the linear estimator of Eq. (26) for majority of signals in practice.

The maximum entropy (ME) estimation is formulated as a following variational problem in time-space domain (rigorous proof can be found elsewhere [4]): maximize the volume intregral

$$\int_{o}^{W} \log S_x(f,\vec{v})d\vec{v} \quad (30)$$

with respect to the frequency-wavenumber spectrum $S_x(f,\vec{v})$, with the constraint that the estimated cross-power spectra $P_x(f,\vec{z}_k,\vec{z}_l)$ for each pair of spatial points $\vec{z}_k$, $\vec{z}_l$ satisfy the inverse Fourier relationships in the wavenumber domain, that is

$$\hat{P}_x(f,\vec{z}_k,\vec{z}_l) = \int_{V_\nu} S_x(f,\vec{v}) \cdot e^{j2\pi\vec{v}\cdot(\vec{z}_k-\vec{z}_l)} d\vec{v}$$

$$l,k = 1, 2, \ldots, K \quad (31)$$

In Eqs. (30) and (31), W is the "cutoff" wavenumber in one dimension, or spatial Nyquist rate, V_ν is the volume encompassing spatial extent $-W_i \leq \nu_i \leq W_i$, i = x, y, z, in the wavenumber space, and the frequency f is constant. It is assumed that the cross-spectrum $P_x(f,\vec{z}_k,\vec{z}_l)$ has already been estimated by some of the methods discussed earlier.

The solution to the variational problem given by Eqs. (30) and (31) is

$$\hat{S}_{x(ME)}(f,\vec{v}) = [\underline{e}^{H}(\vec{v})\underline{\Lambda}(f)\underline{e}(\vec{v})]^{-1} \quad (32)$$

$$\int [\underline{e}^{H}(\vec{v})\underline{\Lambda}(f)\underline{e}(\vec{v})]^{-1}\underline{e}(\vec{v})\underline{e}^{H}(\vec{v})d\vec{v} = \hat{\underline{P}}_x(f) \quad (33)$$

where $\underline{e}(\vec{v})$ is given by Eq. (29) and $\underline{\Lambda}(f)$ is the matrix of Lagrangian multipliers $\lambda_{k,l}$, and $\underline{P}_x(f)$ is the estimated cross-spectral matrix of the sensor signal. The form of the ME estimator given by Eq. (32) is similar to the ML estimator, Eq. (28). It was shown that the ME estimator is superior to the ML estimator in the case of uniformly spaced line array [9]. We note that, in the case of the uniformly spaced line array, the ME estimator can be derived in terms of prediction-error filtering [10].

A large majority of the time series can be modelled by one of three linear models: a moving average (MA), an autoregressive (AR), and a mixed autoregressive-moving average (ARMA) [1]. It has been demonstrated that the MEM works most satisfactorily with signals which can be modelled by an AR process, while the linear methods give the best spectral representation for an MA modelled process. However, by virtue of Wold's decomposition theorem, every process can be represented by an MA process of a sufficiently high order or, equivalently, by an AR process of sufficiently high order. Thus, in principle, we can apply any of the above methods to any time series, providing we use a sufficiently long data record. For practical reasons, we use the method that corresponds to a model with the lower order.

We use the same reasoning in the analysis of time-space series. If the series can be represented by an MA model in both time and space, we apply the linear methods. Similarly, for an AR modelled time-space series, the MEM is a better choice. If the fitting models in time and space are different, it is the spatial model which determines the choice of the method, because of the more severe practical limitations in the number of spatial data samples.

If the time-space series is modelled by an ARMA process, then some ARMA spectral estimation method is used. For example, the response of the uniformly spaced line array to a multitude of plane waves impinging at different angles may be modelled as an ARMA process [11]. Development of ARMA spectral estimation methods for temporal and time-space series is currently an active research field.

Finally, we note that in the cases where the plane wave signals cannot be considered as time and/or space stationary, adaptive methods of signal processing should be used [12].

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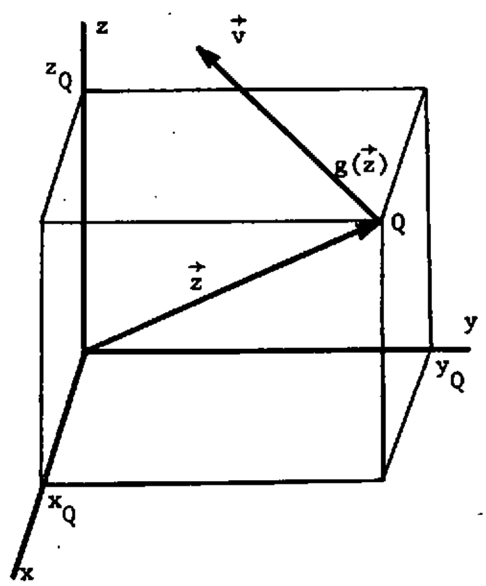

Fig. 1 Three-dimensional representation of the travelling wave $g(\vec{z})$

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