Proceedings of The Institute of Acoustics

PROPAGATION FROM A LINE SOURCE ABOVE AN IMPEDANCE PLANE

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The problem of acoustic propagation from an isotropic monofrequency point source in a homogeneous fluid half-space above a locally reacting plane boundary of uniform surface impedance is one of the basic theoretical problems arising from a study of outdoor sound propagation. Good recent treatments of this problem are given in [15,16,11]. The related, and simpler problem, of propagation from a horizontal coherent line source above a horizontal plane of uniform surface impedance, has received relatively little attention, perhaps because of its less obvious practical application. Exceptions are the discussions of Heins and Feshbach [10], Rasmussen [13], Habault [9], and Chandler-Wilde and Hothersall [2].

Recently the interest in calculation of sound propagation above a homogeneous impedance plane has received a new impetus, due to the development of boundary element methods for the calculation of outdoor sound propagation over inhomogeneous flat terrain, and over noise barriers [14,2,9,3,4,5]. For many practical problems a two-dimensional approximation is adequate. An essential requirement of the two-dimensional boundary element method is a formula for calculating sound propagation from a line source above a homogeneous impedance plane. For the application of such a formula to the boundary element method, the interest is not just in the case when the receiver is a large number of wavelengths from the geometrical image of the source. Image-receiver distances perhaps as small as 1/10 wavelength are also important.

THE SOLUTION IN INTEGRAL FORM

Figure 1 shows the situation in the plane perpendicular to the line source and the Cartesian co-ordinate system used. $\mathbf{r}_0 = (x_0,y_0)$ is the position of the source, $\mathbf{r}_0' = (x_0,-y_0)$ the position of the image of the source, and $\mathbf{r} = (x,y)$ is the position of the receiver.

$$R' = |\mathbf{r} - \mathbf{r}_0'| \qquad (1)$$

is the distance from image to receiver and θ_0 is the angle of incidence.

Let $G_\beta(\mathbf{r},\mathbf{r}_0)$ denote the acoustic pressure at point $\mathbf{r}$ due to a unit simple source at $\mathbf{r}_0$ (time dependence $e^{-i\omega t}$ for the acoustic field is assumed throughout). Then $G_\beta(\mathbf{r},\mathbf{r}_0)$ satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\, G_\beta(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0), \qquad (2)$$

in $y > 0$, the impedance boundary condition,

$$\frac{\partial G_\beta(\mathbf{r},\mathbf{r}_0)}{\partial y} + ik\beta G_\beta(\mathbf{r},\mathbf{r}_0) = 0, \qquad (3)$$

on $y = 0$, and the Sommerfeld radiation condition. In equations (1) and

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(3), k is the wavenumber in the fluid medium,

$\nabla^2 \equiv \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplacian operator, and the constant β is the normalised surface admittance. It will be assumed throughout that either $\beta = 0$ (the plane is rigid) or Re $\beta > 0$ (the plane is energy-absorbing).

In the case when $\beta = 0$, the solution to the above boundary value problem is easily found, by the method of images, to be

$$G_o(r,r_o) = -\frac{i}{4} H_o^{(1)}(kR) - \frac{i}{4} H_o^{(1)}(kR') \tag{4}$$

(where $R = |r-r_o|$). In the general case, it is convenient to write G_β as the sum

$$G_\beta = G_o + P_\beta. \tag{5}$$

By Fourier transform methods, P_β is then determined to be [10,5]

$$P_\beta = \frac{i\beta}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(ik((y+y_o)\sqrt{1-s^2} - (x-x_o)s))}{\sqrt{1-s^2}\,(\sqrt{1-s^2} + \beta)} \, ds \tag{6}$$

where

$$\mathrm{Re}(\sqrt{1-s^2}),\ \mathrm{Im}(\sqrt{1-s^2}) \geqslant 0. \tag{7}$$

To obtain an expression for P_β more suitable for calculation, the substitution $s = \sin\theta$ is made to remove the branch point singularities in the integrand of (6), and the path of integration in the resulting integral is deformed to the steepest descent path. This leads to the representation [13,2,9]

$$P_\beta = P_\beta^{(r)} + P_\beta^{(s)} \tag{8}$$

where

$$P_\beta^{(r)} = \frac{\beta e^{i\rho}}{\pi} \int_0^\infty t^{-1/2} e^{-\rho t} f(t)\, dt, \tag{9}$$

$$\rho = kR', \quad \gamma = \cos\theta_o, \tag{10,11}$$

$$f(t) = -\frac{(\beta + (1+it)\gamma)}{\sqrt{t-2i}\,(t-ia_+)(t-ia_-)}, \qquad \mathrm{Re}(\sqrt{t-2i}\ ) > 0, \tag{12}$$

$$= -\frac{(\beta + (1+it)\gamma)}{\sqrt{t-2i}\,(t^2-2i(1+\beta\gamma)t - (\beta+\gamma)^2)}, \tag{13}$$

$$a_\pm = 1 + \beta\gamma \mp \sqrt{1-\beta^2}\ \sqrt{1-\gamma^2}\ , \quad \mathrm{Re}\ (\sqrt{1-\beta^2}) \geqslant 0, \tag{14}$$

$$P_\beta^{(s)} = \begin{cases} (\beta/\sqrt{1-\beta^2})\, e^{i\rho(1-a_+)}, & \text{if Im } \beta < 0 \text{ and Re } a_+ < 0, \\ \tfrac{1}{2}(\beta/\sqrt{1-\beta^2})\, e^{i\rho(1-a_+)}, & \text{if Im } \beta < 0 \text{ and Re } a_+ = 0, \\ 0\,, & \text{otherwise.} \end{cases} \qquad (15)$$

Provided the function $f(t)$ is a smooth, slowly-varying function on the positive real axis, the representation (9) for $P_\beta^{(r)}$ is very suitable for numerical integration. In fact Gaussian quadrature can be applied with weight function $t^{-\frac{1}{2}} e^{-\rho t}$. To apply Gaussian quadrature it is convenient to rewrite $P_\beta^{(r)}$ in the form

$$P_\beta^{(r)} = \frac{\beta e^{i\rho}}{\pi\sqrt{\rho}} \int_0^\infty s^{-\frac{1}{2}} e^{-s} f(s/\rho)\, ds. \qquad (16)$$

It is easy to see that, for all values of β and γ,

$$\text{Re } a_- \geq 1. \qquad (17)$$

Also

$$\text{Re } a_+ \geq 1 - |1-\beta^2|^{\frac{1}{2}} > \tfrac{1}{4} \quad \text{if } |1-\beta| < \tfrac{1}{4}. \qquad (18)$$

Thus, if $|1-\beta| < \frac{1}{4}$, $f(t)$ is a regular analytic function in Im $t < \frac{1}{4}$. Note also that, from (15) and (18),

$$P_\beta = P_\beta^{(r)}, \text{ if Im } \beta \geq 0 \text{ or } |1-\beta| < \tfrac{1}{4}. \qquad (19)$$

Unfortunately, when β and γ are small (a frequent combination in outdoor sound propagation)

$$a_+ \approx \tfrac{1}{2}(\beta + \gamma)^2 \qquad (20)$$

and ia_+ lies near the positive real axis. Equation (9) is not so suitable for calculation when this is the case.

To obtain an alternative representation for P_β, the pole of $f(t)$ at $t = ia_+$, which causes the difficulty, is subtracted. A function g, which can be shown to be regular in Im $t < 1$ [5], is defined by

$$g(t) = f(t) - \frac{e^{-i\pi/4}\sqrt{a_+}(t-ia_+)^{-1}}{2\sqrt{1-\beta^2}}, \qquad \text{Re}(\sqrt{a_+}) > 0. \qquad (21)$$

Substituting for f in (16), using equation (19), and a representation of the error function [1, equation (7.1.3)], it is found that, for Im $\beta > 0$,

$$P_\beta = \frac{\beta e^{i\rho}}{\pi\sqrt{\rho}} \int_0^\infty s^{-\frac{1}{2}} e^{-s} g(s/\rho) ds + \frac{\beta e^{i\rho(1-a_+)}}{2\sqrt{1-\beta^2}} \text{erfc}(e^{-i\pi/4}\sqrt{\rho}\,\sqrt{a_+}). \qquad (22)$$

It can be shown [5] that, with the choice of branch cuts made in (14) and

(21), the right hand side of this equation, considered as a function of β, is regular in the half-plane Re $\beta > 0$, cut from 1 to $+\infty$ along the real axis. From equation (6) it is clear that P_β is regular throughout the half-plane Re $\beta > 0$. Hence, by analytic continuation and continuity, equation (22) holds throughout Re $\beta > 0$, except clearly at $\beta = 1$.

Equations (16) and (22) do not apply if $\rho = 0$, and this is an important case for the application to the boundary element method. Fortunately, when $\rho = 0$, the integration can be performed analytically [5], giving

$$P_\beta(r,r) = \begin{cases} \dfrac{i\beta}{2\pi\sqrt{1-\beta^2}} \ln\left(\dfrac{\beta - i\sqrt{1-\beta^2}}{\beta + i\sqrt{1-\beta^2}}\right), & \beta \neq 1, \\ \dfrac{1}{\pi}, & \beta = 1, \end{cases} \qquad (23)$$

where the principal value of the logarithm is to be taken.

THE USE OF GAUSS-LAGUERRE QUADRATURE

The representations (16) and (22) are in the correct form for the application of Gauss-Laguerre quadrature [7]. Thus the following approximation for P_β can be proposed. From equations (16) and (19), for $|1-\beta| < \frac{1}{4}$,

$$P_\beta \approx \tilde{P}_{n,m} := \frac{\beta e^{i\rho}}{\pi\sqrt{\rho}} \sum_{j=1}^{m} w_{j,n}\, f(x_{j,n}/\rho), \qquad (24)$$

where $n \geq 1$, $1 \leq m \leq n$, and $w_{1,n},\ldots, w_{n,n}$ are the weights and $x_{1,n},\ldots, x_{n,n}$ are the abscissae of the n-point Gauss-Laguerre quadrature rule with weight function $s^{-\frac{1}{2}} e^{-s}$. Values of these abscissae and weights are tabulated for $n = 1, 2, \ldots, 15$ in columns 1 and 2 respectively of Table II in Concus et al [7]. Note that $x_{1,1} = \frac{1}{2}$ and $w_{1,1} = \sqrt{\pi}$, and that only when $m = n$ in (24) is the full Gauss-Laguerre quadrature rule being used.

By applying Gauss-Laguerre quadrature to (22) an alternative approximation is obtained, for $\beta \neq 1$, namely

$$P_\beta \approx \hat{P}_{n,m} := \frac{\beta e^{i\rho(1-a_+)}}{2\sqrt{1-\beta^2}} \operatorname{erfc}(e^{-i\pi/4}\sqrt{\rho}\,\sqrt{a_+}) + \frac{\beta e^{i\rho}}{\pi\sqrt{\rho}} \sum_{j=1}^{m} w_{j,n}\, g(x_{j,n}/\rho). \qquad (25)$$

Note that the calculation of the complementary error function erfc is discussed in [12,6], and coefficients of Padé approximants for erfc are given in Table 2.1 of [5].

It can be shown that f is analytic and bounded in a region including the real axis if $|1-\beta| < \frac{1}{4}$, while g is analytic and bounded in a region including the real axis if $|\beta|$ is bounded and $|1-\beta|$ is bounded away from zero. In fact it can be shown that [5]

$$|f(t)| < 25, \quad \text{for } |1-\beta| < \frac{1}{4} \text{ and } \mathrm{Im}\, t \leq \frac{1}{8} \;, \tag{26}$$

and similar arguments establish that

$$|g(t)| < 90 + 4|1-\beta|^{-1/2}\,, \text{ for } |\beta| \leq 1 \text{ and } \mathrm{Im}\, t \leq 1/4. \tag{27}$$

This feature makes the approximations (24) and (25) complementary; (24) is suitable for β near 1 and (25) suitable elsewhere.

The remainder of this section considers the errors made in using the approximations (24) and (25).

$$|P_\beta-\tilde{P}_{n,m}| \leq |P_\beta-\tilde{P}_{n,n}| + |\tilde{P}_{n,n}-\tilde{P}_{n,m}| \tag{28}$$

and, from (26),

$$|\tilde{P}_{n,n}-\tilde{P}_{n,m}| < 10\rho^{-1/2} \sum_{j=m+1}^{n} w_{j,n} \;. \tag{29}$$

From (26) and standard results on Gauss-Laguerre quadrature [7,8], it can be shown that [5]

$$|P_\beta-\tilde{P}_{n,n}| < 71\ (2n)!2^{4n-3}\ \rho^{-(2n+1/2)},\ \text{for } |1-\beta| < 1/4, \tag{30}$$

and that, for all α, $\rho_0 > 0$,

$$|P_\beta-\tilde{P}_{n,n}| \leq c_n\ \rho^{-\alpha},\ \text{for } |1-\beta| < 1/4,\ \rho \geq \rho_0, \tag{31}$$

where $c_n \to 0$ as $n \to \infty$. For each n, c_n depends on n, α, and ρ_0, but is independent of ρ, β, and γ.

Similarly

$$|P_\beta-\hat{P}_{n,m}| \leq |P_\beta-\hat{P}_{n,n}| + |\hat{P}_{n,n} - \hat{P}_{n,m}|, \tag{32}$$

$$|\hat{P}_{n,n}-\hat{P}_{n,m}| < |\beta|\pi^{-1}\ \rho^{-1/2}(90+4|1-\beta|^{-1/2}) \sum_{j=m+1}^{n} w_{j,n},\ \text{for } |\beta| \leq 1, \tag{33}$$

$$|P_\beta-\hat{P}_{n,n}| < (45+2|1-\beta|^{-1/2})\pi^{-1/2}(2n)!4^n\rho^{-(2n+1/2)},\ \text{for } |\beta| \leq 1, \tag{34}$$

and, for all α, ρ_0, $\epsilon > 0$,

$$|P_\beta-\hat{P}_{n,n}| \leq d_n\ \rho^{-\alpha},\ \text{for } \rho \geq \rho_0,\ \ |1-\beta| \geq \epsilon,\ |\beta| \leq 1, \tag{35}$$

where $d_n \to 0$ as $n \to \infty$ with α, ρ_0 and ϵ fixed.

The above bounds show clearly that the approximation (24) is more appropriate for β near 1, and (25) more appropriate elsewhere. Note also that (30) and (34) show that $\tilde{P}_{n,n}$ and $\hat{P}_{n,n}$ are excellent approximations in the far-field ($\rho \to \infty$) even if only one quadrature point is used (n=1). The sum of weights which occurs in the bounds (29) and (33) is usually small if both m and n are large. Since the work in evaluating the sums in (24) and (25) is proportional to m, it is desirable to keep m as low as possible.

In a FORTRAN subroutine written by the authors the approximation $\hat{P}_{40,22}$ has been adopted for $|1-\beta| \geq 0.1$ and $\hat{P}_{40,22}$ for $|1-\beta| < 0.1$. For $\hat{P}_{40,22}$ an interesting quantity is

$$|P_\beta-\hat{P}_{40,22}|/|-(i/4)H_o^{(1)}(\rho)|. \qquad (36)$$

This is the error which will arise in the reflection coefficient if it is calculated using $\hat{P}_{40,22}$ as an approximation for P_β. Estimates of the maximum value of the quantity (36) when ρ is fixed, β varies in the range, $|\beta| \leq 0.8$, $-69° \leq \arg \beta \leq 89°$, and either $\gamma = 0$ or $0 \leq \gamma \leq 1$, are shown, for values of ρ, in Table 1. (To calculate these estimates P_β is replaced by $P_{100,100}$ in (36), and (36) is then evaluated for $|\beta| = 0.1(0.1)0.8$, $\arg \beta = -69°(9.9°)89°$, $\gamma = \cos\theta_o$, $\theta_o = 0°(10°)90°$.) Note that the range of β considered includes most values of interest in outdoor sound propagation. Since

$$\sum_{j=23}^{40} w_{j,n} < 1.9 \times 10^{-15} \qquad (37)$$

and

$$|-(i/4)H_o^{(1)}(\rho)| > 0.25\pi^{-1/2}\rho^{-1/2}, \text{ for } \rho > 1/2, \qquad (38)$$

it follows from (32) and (33) that, for $|\beta| \leq 0.8$ and $\rho > 1/2$,

$$|P_\beta-\hat{P}_{40,22}|/|-(i/4)H_o^{(1)}(\rho)| < |P_\beta-\hat{P}_{40,40}|/|-(i/4)H_o^{(1)}(\rho)| + 3.4\times10^{-13}. \qquad (39)$$

For values of ρ greater than those shown in Table 1, the maximum value of the error (36) is found to remain at about 8×10^{-13}, which, taking into account rounding errors, is consistent with the bounds (35) and (39).

SOME FURTHER FORMULAE

The previous sections have shown how G_β can be calculated accurately. In the two-dimensional boundary integral equation method for the calculation of sound propagation from a line source over a noise barrier [4] it is important also to be able to calculate the derivatives of G_β. From (4) and (5), since

$$H_o^{(1)'}(x) = -H_1^{(1)}(x),$$

$$\nabla G_\beta(r,r_o) = \frac{ik}{4} H_1^{(1)}(kR)\frac{(r-r_o)}{R} + \frac{ik}{4} H_1^{(1)}(kR')\frac{(r-r_o')}{R'} + \nabla P_\beta(r,r_o). \qquad (40)$$

It follows, basically from the boundary condition (3), that

$$\frac{\partial}{\partial y} P_\beta(r,r_o) = \frac{-k\beta}{2} H_o^{(1)}(\rho) - ik\beta\, P_\beta(r,r_o), \qquad (41)$$

while an argument along the lines of that leading to (22) shows that

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$$\frac{\partial}{\partial x} P_\beta(r,r_0) = ik\beta\, e^{i\rho}\, \mathrm{sign}(x-x_0)\Big(\tfrac{1}{2}e^{-i\rho a_+}\, \mathrm{erfc}(e^{-i\pi/4}\sqrt{\rho}\,\sqrt{a_+}) - \pi^{-1}\rho^{-\frac{1}{2}}\int_0^\infty s^{-\frac{1}{2}}\, e^{-s} g^*(s/\rho)ds\Big) \qquad (42)$$

where

$$g^*(t) = \frac{\sqrt{1-\gamma^2}\ (\beta+\gamma+i\beta t)}{\sqrt{t-2i}\ (t-ia_+)(t-ia_-)} - \frac{e^{-i\pi/4}\sqrt{a_+}}{2(t-ia_+)} . \qquad (43)$$

The Gauss-Laguerre quadrature of the previous section can be used to calculate $\partial P_\beta/\partial x$, and the similarities between the expressions (22) and (42) make it efficient to calculate P_β and its derivatives simultaneously.

An analysis similar to that leading to equations (16) and (22) can be given for the three-dimensional problem of propagation from a point source over a plane of admittance β [5]. (For this problem Figure 1 should be thought of as showing a vertical cross-section through point source and receiver). In the general case the expressions corresponding to (16) and (22) are somewhat complex. Some simplification occurs if $\gamma = 0$ (i.e. at grazing incidence). G_β (which satisfies (2) and (3) as in the 2-D case, though with ∇^2 and δ now the 3-D Laplacian and Dirac delta function respectively) satisfies equation (5) with (if $\gamma=0$),

$$G_0 = -e^{i\rho}/(2\pi R), \qquad (44)$$

$$P_\beta = \begin{cases} \tfrac{1}{2}k\beta\, e^{i\rho}\pi^{-1}\rho^{-\frac{1}{2}} \displaystyle\int_0^\infty s^{-\frac{1}{2}}e^{-s}\hat{f}(s/\rho)ds, & |1-\beta| < \tfrac{1}{4}, \\ \tfrac{1}{2}k\beta\, e^{i\rho}\pi^{-1}\rho^{-\frac{1}{2}} \displaystyle\int_0^\infty s^{-\frac{1}{2}}e^{-s}\hat{g}(s/\rho)ds + \frac{k\beta}{4} H_0^{(1)}(\rho\sqrt{1-\beta^2})\mathrm{erfc}(e^{-i\pi/4}\sqrt{\rho}\ \sqrt{a_+}), & \beta\neq 1, \end{cases} \qquad (45a,b)$$

where

$$\hat{f}(t) = \frac{-\beta e^{\rho(t-i)}H_0^{(1)}(\rho(1+it))(1+it)}{\sqrt{t-2i}\ (t-ia_+)(t-ia_-)} , \qquad (46)$$

$$\hat{g}(t) = \hat{f}(t) - \tfrac{1}{2}e^{-i\pi/4}\sqrt{a_+}\ e^{-i\rho\sqrt{1-\beta^2}}\ H_0^{(1)}(\rho\sqrt{1-\beta^2})(t-ia_+)^{-1}. \qquad (47)$$

$\hat{g}(t)$ is regular in $\mathrm{Im}\ t < 1$, and, if $|1-\beta| < \frac{1}{4}$, $\hat{f}(t)$ is regular in $\mathrm{Im}\ t < \frac{1}{4}$. The Gauss-Laguerre methods of the previous section should be used to evaluate the integrals in (45). The representations (45) may prove useful for the numerical solution of the boundary integral equation [9,5] describing propagation from a point source above an inhomogeneous impedance plane.

CONCLUSIONS

Accurate and efficient approximations (equations (4), (5), (24) and (25)) for calculation of sound propagation from a line source above a homogeneous impedance plane have been described and analysed. The

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approximation based on equation (25), with $n = 40$ and $m = 22$, has been shown, by theoretical analysis and systematic calculations, to be accurate for all angles of incidence, and for a range of surface admittance which includes those values usual in outdoor sound propagation. High accuracy is retained even when the distance between image and receiver is as small as 1/10 wavelength.

Similar approximations have been outlined for calculating the spatial derivatives of the solution for propagation from a line source, and for the problem of propagation from a point source over a homogeneous impedance plane at grazing incidence.

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Table 1. The estimated maximum error, over the ranges of β and γ indicated, in the reflection coefficient, when it is calculated by using $\hat{P}_{40,20}$ as an approximation for P_β.

ρ	$\rho/(2\pi)$	$\lvert\beta\rvert < 0.8$, $-69^\circ < \arg\beta < 89^\circ$	
		$0 < \gamma < 1$	$\gamma = 0$
0.5	0.0796	3.2×10^{-5}	7.7×10^{-7}
0.75	0.119	2.8×10^{-6}	2.8×10^{-8}
1.125	0.179	1.3×10^{-7}	4.5×10^{-10}
1.688	0.269	2.7×10^{-9}	3.4×10^{-12}
2.531	0.403	2.1×10^{-11}	7.9×10^{-13}
3.797	0.604	8.0×10^{-13}	8.0×10^{-13}

Figure 1. Geometry of source and receiver above impedance plane