HARMONIC RADIATIONS IN THE FIELD OF A BAFFLED PISTON

by

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ABSTRACT

When the radiation from a baffled piston is sufficiently intense, a number of novel and perhaps useful phenomena related to harmonic generation appear, as a result of nonlinear interactions in the sound field. This paper reports some recent studies of several of the more prominent and interesting of these phenomena; in particular, shock formation, finite amplitude attenuation and the propagation and directivity of harmonic radiations. Theory pertinent to the delineation of acoustic fields is given, along with the results of several experiments done in a fresh water lake.

Introduction

One of the most popular problems in acoustics is the radiation of sound from a piston source. The importance of this problem has to do with its relevance to practical applications such as the radiation of sound from loud speakers and from sonar transducers. Up to the present time, textbook descriptions of the piston problem have invariably treated only the linear or first order field. This has been the case, despite the fact that significant work related to second order effects in traveling waves generated by piston sources dates back to Earnshaw**(1860).

Many recent investigations have improved our understanding of not only the second order field but also its effect on the primary radiation. Notable among these investigations is the experimental work of Ryan, Lutsch and Beyer[2](1961), which treated the distortion

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**For a history of the subject, see reference 1.

of rf pulses in water. The results of their measurements, made in the nearfield of a quartz projector with a relatively large hydrophone, were compared to the perturbation theory of Keck and Beyer[3] (1960). The growth of harmonics was found to increase with range in agreement with plane wave theory. Gould, Smith, Williams and Ryan[4] (1965) extended this work in some nearfield measurements of the fundamental, second and third harmonics, using a tiny probe hydrophone. They found a considerable fine structure in both the angular and the axial response for all components. The Brown University group has also studied the nearfield theoretically; see papers by Ingenito, Williams, and Rogers.[5]

Although the nearfield of the piston is an important region, it is quite often not as important as the farfield region. This is especially true in systems of practical significance in sonar. The farfield region of the piston is characterized by spherical divergence and Bessel directivity. Although much work has been done on spherical waves of finite amplitude,[6-12] very little has been done in the area of nonuniform spherical waves associated with radiations from piston transducers.

In the present paper, we attempt a survey of the piston problem with emphasis on a more extensive delineation of several second order entities arising from farfield interaction. The classic problem of distortion and shock formation is the first to be examined. Certain approximate modifications to weak shock theory are made that enable its results to be extended to problems involving long range propagation, where ordinary exponential damping eventually dominates. These modifications will be compared to the results of several hydroacoustic experiments on the propagation and directivity of spherical waves of finite amplitude.

Weak Shock Theory

The key steps in the progressive distortion and eventual formation of shock waves are sketched in Fig. 1. It is assumed that we start with a high amplitude sine wave at some range r_o, which may be taken as the range of spherical divergence. As is well known, the dependence of the signal speed on location within the waveform causes a cumulative distortion until a shock wave is formed at a range $\bar{r}$. At this range, the amplitude of the fundamental is reduced by 1 dB as a result of the distortion that has taken place. If the wave is of sufficient amplitude, it attains a mature sawtooth shape at the range $\tilde{r}$. At this range, the amplitude of the fundamental is reduced by 6 dB. Finally, dissipation reduces the shock wave to an "old age" waveform that is basically sinusoidal. The waveforms shown in this figure were constructed by use of weak shock theory and are applicable to spherical waves of finite amplitude. For example, successive positions of points on the waveform were determined from

$$t = \Phi + \int_{r_o}^{r} \frac{dr}{v} , \qquad (1a)$$

where

$$\frac{1}{v} = \frac{1}{c_o + \beta u r_o / r} \doteq \frac{1}{c_o} - \frac{\beta u r_o}{c_o^2 r} , \qquad (1b)$$

$$\Phi = \sin^{-1}(u/u_o) ,$$

and

$$\beta = 1 + \frac{B}{2A} .$$

Here c_o is the small-signal sound speed and B/A is the parameter of nonlinearity. When shocks form, u is replaced in Eq. (1b) by $(u_a + u_b)/2$, where u_a is the value of u just ahead of the shock while u_b is the value of u just behind the shock. The formulas for $\bar{\bar{r}}$ and $\tilde{r}$ shown in Figure 1 were also derived from the weak shock solution (see below). The symbols are defined as follows:

$\varepsilon = \frac{u_o}{c_o}$, the acoustic Mach number, u_o being the peak particle velocity at $r = r_o$, and

k = the wavenumber of the fundamental component, $\frac{2\pi}{\lambda}$,

r_{max} = the range for which the rate of finite amplitude attenuation is equal to that of small-signal attenuation for the fundamental

and

α = small signal attenuation coefficient.

In the frequency domain, the weak shock solution may be expressed in terms of a Fourier series as follows:[13]

$$p = p_o\left(\frac{r_o}{r}\right) \sum_{n=1}^{\infty} B_n \sin n\left[\omega t - k(r-r_o)\right] , \qquad (2)$$

where

$$p_o = \rho_o c_o u_o \quad , \quad \text{and}$$

$$B_n = \frac{2}{n\pi} W_b + \frac{2}{n\pi\sigma}\int_{\Phi_{min}}^{\pi} \cos n\,(\Phi - \sigma \sin\Phi)\,d\Phi \quad ,$$

$$W_b = \left(\frac{1}{\sigma}\right)\left[j_o^{-1}\left(\frac{1}{\sigma}\right)\right] U(\sigma - 1) \quad ,$$

$$\Phi_{min} = \sigma W_b ,$$

$$\sigma = \beta\varepsilon k r_o \ln\frac{r}{r_o} .$$

For $j_o^{-1}\left(\frac{1}{\sigma}\right)$ read, "the quantity whose zero order spherical Bessel function is $\frac{1}{\sigma}$." The symbol $U(\cdot)$ stands for the unit step function.

The frequency domain solution allows the calculation of the amplitudes of the various frequency components. [Similar solutions for various segments of the propagation curve have been obtained by Naugol'nykh, Soluyan, and Khokhlov[9] (1963).] The B_n's in the weak shock solution are the coefficients of the harmonic components of the waveform. When the distortion parameter, σ, is less than unity, the B_n are analogous to those obtained by Fubini[14] for plane waves in the shock free region. When $\sigma > 3$, the limiting forms for B_n may be associated with the sawtooth solution, which is an asymptotic form of the Fay[15] solution for $\beta\varepsilon k >> \alpha$.

Dissipation Considerations

A difficulty associated with the weak shock solution is that it is not valid at ranges concomitant with significant small signal attenuation. Blackstock[13] estimated the range of validity of the weak shock solution by determining the range at which the rates of finite amplitude and small signal attenuation are equal for the fundamental component. We can extend this computation to the nth harmonic by imposing the condition,

$$\left.\frac{dB_n(r)}{dr}\right|_{r=r_{max}} = B_n(r_{max}) \left.\frac{de^{-\alpha_n(r-r_{max})}}{dr}\right|_{r=r_{max}} , \qquad (3)$$

where B_n is given by the weak shock solution. If the wave is strong enough for a mature sawtooth to form, $B_n = 2/n(1+\sigma)$, in which case Eq. (3) gives

$$r_{max}^{(n)} = \frac{\beta\epsilon k r_o}{\alpha_n\left[1+\beta\epsilon k r_o \ln \frac{r_{max}^{(n)}}{r_o}\right]} . \qquad (4)$$

It should be remarked that extension of the r_{max} criterion to the n^{th} harmonic is not valid unless the component in question is already in a state of finite amplitude decay. At ranges shorter than $r_{max}^{(n)}$, the weak shock solution is reasonably valid, at least for the harmonic component in question. At greater ranges, one must resort to the Burgers' equation, a nonlinear equation for progressive waves that includes the effects of viscosity.* An exact solution of Burgers' equation is known for plane waves. For spherical waves, however, no complete, uniformly valid solutions are known. On the other hand, partial solutions valid for certain ranges are known** and some promising numerical solutions have also been developed and are discussed elsewhere in these proceedings.

As an alternate approach, we seek to modify weak shock theory to obtain an approximate formula that will offer a simpler solution to the problem. We begin by considering the two agencies of dissipation, i.e., finite amplitude attenuation and small signal absorption, for the fundamental component, as shown graphically in Fig. 2, part (a). It can be seen that finite-amplitude attenuation dominates at ranges shorter than the equal-rate range, $r_{max}^{(1)}$. At ranges in excess of $r_{max}^{(1)}$, small-signal absorption dominates. Since the attenuation rates of the two agencies are predicted to be the same at $r_{max}^{(1)}$, we choose an approximate Fourier coefficient that is given by weak-shock theory for ranges shorter than $r_{max}^{(1)}$ and makes a smooth transition to small-signal attenuation at $r_{max}^{(1)}$. This approach may be generalized, and the Fourier coefficient for the nth harmonic is given by

$$A_n(\beta,\epsilon,k,r) = \left\{\left[1 - U\left(r-r_{max}^{(n)}\right)\right]B_n(\beta,\epsilon,k,r) + \left[U\left(r-r_{max}^{(n)}\right)\right]B_n\left(\beta,\epsilon,k,r=r_{max}^{(n)}\right)e^{-\alpha_n\left(r-r_{max}^{(n)}\right)}\right\} \qquad (5)$$

*See, for example, Ref. 16.

**See, for example, Ref. 9.

Here the unit step function, $U(\cdot)$, is used to indicate the transition from one form of solution to the other. That the transition is smooth is evident from the plot for the fundamental component shown in Fig. 2, part (b).

TYPE WITHIN THE RULED AREAS

Propagation Measurements

The extent to which this model may be valid is demonstrated by a comparison with the experimental results shown in Fig. 3. A baffled, lead-zirconate, lead-titanate piston, 3 inches in diameter, was placed at a depth of 10 ft in a fresh water lake and driven at a source level of 127 dB re 1 μbar at 1 yd at a frequency of 450 kHz. The water column was isothermal, with a temperature of 55°F. The amplitudes of the fundamental and the second and third harmonics were then measured as a function of range. The approximate Fourier coefficient, A_n, can be seen to fall reasonably close to the experimental data points, thereby demonstrating its utility. It has been our experience, however, that this model works best for the fundamental (n=1) component. Additional experimental evaluations of the Fourier coefficient given by Eq. (5) (for the fundamental component) are given in reference 17. A major problem in describing finite amplitude propagation in the farfield is the difficulty of taking into account finite amplitude effects in the nearfield of an electroacoustic source. We attempted to minimize this difficulty by driving a small transducer at high amplitudes so that most of the nonlinear effects would occur in the farfield of the source. Our calculations were made with a value of 1 yd for r_o, which we assumed was the point at which spherical spreading began, and we ignored nearfield distortion. Two excellent papers that analyze the nearfield of a circular piston using linear theory have recently appeared, one by Zemanek[18] (1970), the other by Hobaek[19] (1971). Perhaps the results of their studies, as well as those cited in ref. 5, can be used to advantage in future calculations.

Directivity of Harmonics

The directivity of the harmonic components in spherical waves of finite amplitude can be obtained by several procedures, among which are the method of Westervelt and Radue[20] (1961) as well as methods involving the weak-shock solution. The latter methods have been pursued by the authors of this paper[21-22]. We begin by modifying the boundary condition in the derivation of the weak shock solution by including the usual Bessel directivity factor of a piston source. The new boundary condition is

$$p = p_o \frac{r_o}{r} D(\theta) \sin \omega t \quad , \tag{6}$$

where

$$D(\theta) = \frac{2J_1(ka \sin \theta)}{(ka \sin \theta)}$$

It can easily be shown that the new weak shock solution becomes

$$p = p_o \frac{r_o}{r} D(\theta) \sum_{n=1}^{\infty} B_n(\sigma) \sin n\left[\omega t - k(r-r_o)\right] , \tag{7}$$

where

$$\sigma = \beta \epsilon D(\theta) k r_o \ln\left(\frac{r}{r_o}\right)$$

In effect, this is equivalent to replacing the acoustic Mach number ϵ by $\epsilon D(\theta)$. Then any known solution for uniform spherical waves, i.e., the Fubini solution, the sawtooth solution or the weak shock solution can then be used for directive waves.

The directivity function of the nth harmonic can be obtained by normalization with respect to the amplitude at $\theta=0$, i.e.,

$$D_n(\theta) = \frac{P_n(\theta)}{P_n(0)} \quad . \qquad (8)$$

Two particularly interesting limiting cases are as follows:

I. For $\sigma < 1$

$$D_n(\theta) = \frac{J_n[n\sigma_1 D(\theta)]}{J_n[n\sigma_1]} \quad , \qquad (9a)$$

where σ_1 is the value of σ when $D(\theta)=1$.
If $n\sigma_1$ is small enough for the first term in the series expansion of each Bessel function to be a good approximation of the Bessel function, the result is exceedingly simple, namely

$$D_n(\theta) = D^n(\theta) \quad . \qquad (9b)$$

II. For $\sigma > 3$, $r < r_{max}^{(n)}$,

$$D_n(\theta) = D(\theta)\frac{1+\sigma_1}{1+\sigma_1 D(\theta)} \quad . \qquad (10)$$

The first case is applicable before shocks form. Equation (9b) shows that for relatively mild distortion the directivity of the nth harmonic is equal to the directivity of the fundamental to the nth power. This result was obtained previously by Westervelt and Radue for the special case of the second harmonic.[20] The second case, for strong waves at short ranges, results in an expression for the directivity that is independent of harmonic number. This result is valid only near the center of the major lobe when the wave there is a sawtooth. In this region, the beam pattern of each frequency component becomes blunted and the suppression of the minor lobes is reduced. This effect has been demonstrated experimentally for the fundamental component by Shooter, et al[23] (1970).*

A comparison of theoretical and experimental data on the directivity of the first three components for case I ($\sigma < 1$) is shown in Fig. 4. (The experimental data was taken with the aforementioned 3 in. piston source in conjunction with the propagation measurements presented in the previous figure.) The data shown are for a fundamental source level of 110 dB re 1 μbar at 1 yd at a measurement range of 117 yd. The agreement between theory and experiment is quite good. It can be seen that the beamwidths are decreased with increasing harmonic number. It can also be seen that theory predicts a reduction in the levels of the minor lobes with increase in harmonic number.

A presentation of experimental data on the first three components for higher intensity waves is shown in Fig. 5. Here, the source level was 127 dB re 1 μbar at 1 yd, which is considerably higher than that of

*See also reference 17.

the previous case (L_s = 110 dB) but not quite high enough for comparison to the theory of the second limiting case, i.e., $\sigma > 3$, $r < r_{max}^{(n)}$. For this reason, only the experimental data are shown. One can nonetheless see that the beamwidth of each component has increased as a result of finite amplitude attenuation.

A curious effect is noted in the experimental data for the second harmonic beam pattern. A splitting in the first minor lobe has occurred that is not predicted by theory. This effect has also been observed in the measured patterns of second harmonic radiations in other experiments we have done, with other sources, at different frequencies. One irreverent observer has suggested that the phonons in this region have opposite spins, thereby giving rise to the doublet! (The correct explanation probably has to do with distortion at the transducer face or in the nearfield.)

Discussion

What is the practical significance of shock formation and the attendant generation of harmonics? As is the case with just about any phenomenon, there are some advantages and disadvantages, depending on the potential application in question. The finite amplitude attenuation associated with this process can be a most undesirable effect in sonar since it causes an extra attenuation that may reduce the effective range. On the other hand, nonlinearly induced attenuation would be a welcome effect were it to play a role in the reduction of noise from sources such as jet engines. Solutions such as those presented here may be of value in the prediction of such effects.

We have already seen that at least up to the point of shock formation, and perhaps somewhat beyond, each successive harmonic has a narrower beam pattern and increased suppression of minor lobes. Each of these trends is most desirable in high resolution sonar. The question remains, however, whether it is practical to use the harmonics when their amplitudes must be low in comparison with the fundamental. Taking the range $r = \bar{r}$ as a typical point where the beam patterns of the harmonics still have the desirable properties, we find the second harmonic to be down 8 dB with respect to the fundamental, the third harmonic down 13 dB, the fourth harmonic down 16 dB, and so on. These reductions in level may not be prohibitive for some systems, depending on the tradeoffs involved.* Other possibilities exist.

Acknowledgements

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*Marsh, Mellen and Konrad (private communication) have experimented with systems of this type.

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INITIAL WAVEFORM

$$r = r_0$$

SHOCK FORMATION

$$r = \bar{r} = r_0 \exp\left(\frac{1}{\beta \epsilon k r_0}\right)$$

MATURE SAWTOOTH

$$r = \tilde{r} = r_0 \exp\left(\frac{3}{\beta \epsilon k r_0}\right)$$

OLD AGE

$$r > r_{max} = \frac{\beta \epsilon k r_0}{\alpha\left[1 + \beta \epsilon k r_0 \ln\left(\frac{r_{max}}{r_0}\right)\right]}$$

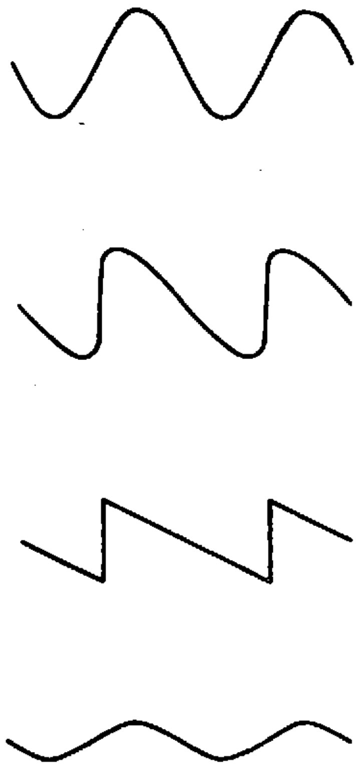
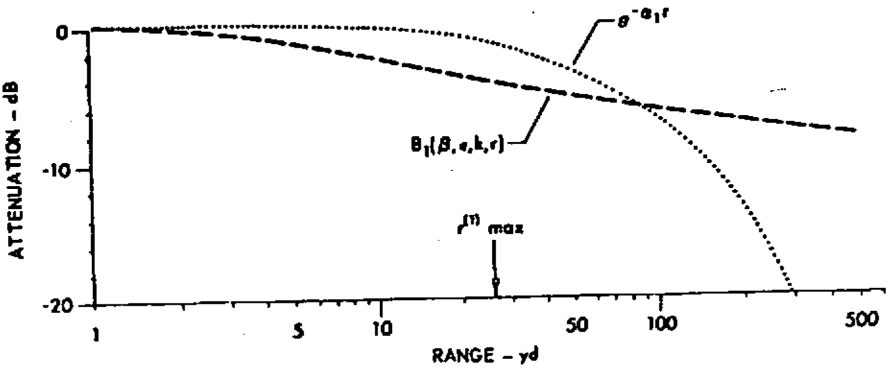
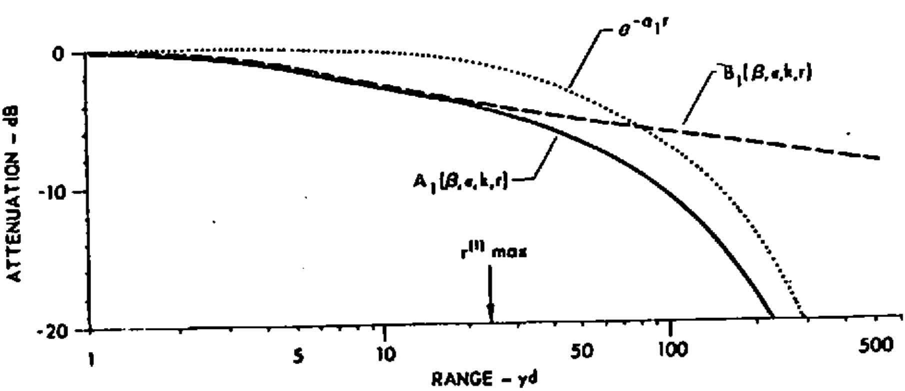


FIGURE 1
CASE HISTORY OF A
FINITE AMPLITUDE WAVEFORM

ARL - UT
AS-71-394
TGM - RFO
5 - 26 - 71



(a) AGENCIES OF DISSIPATION



(b) COMPARISON OF ATTENUATION FACTORS

FIGURE 2
ATTENUATION CONSIDERATIONS

ARL - UT
AS-71-686
TGM - RFO
5 - 26 - 71

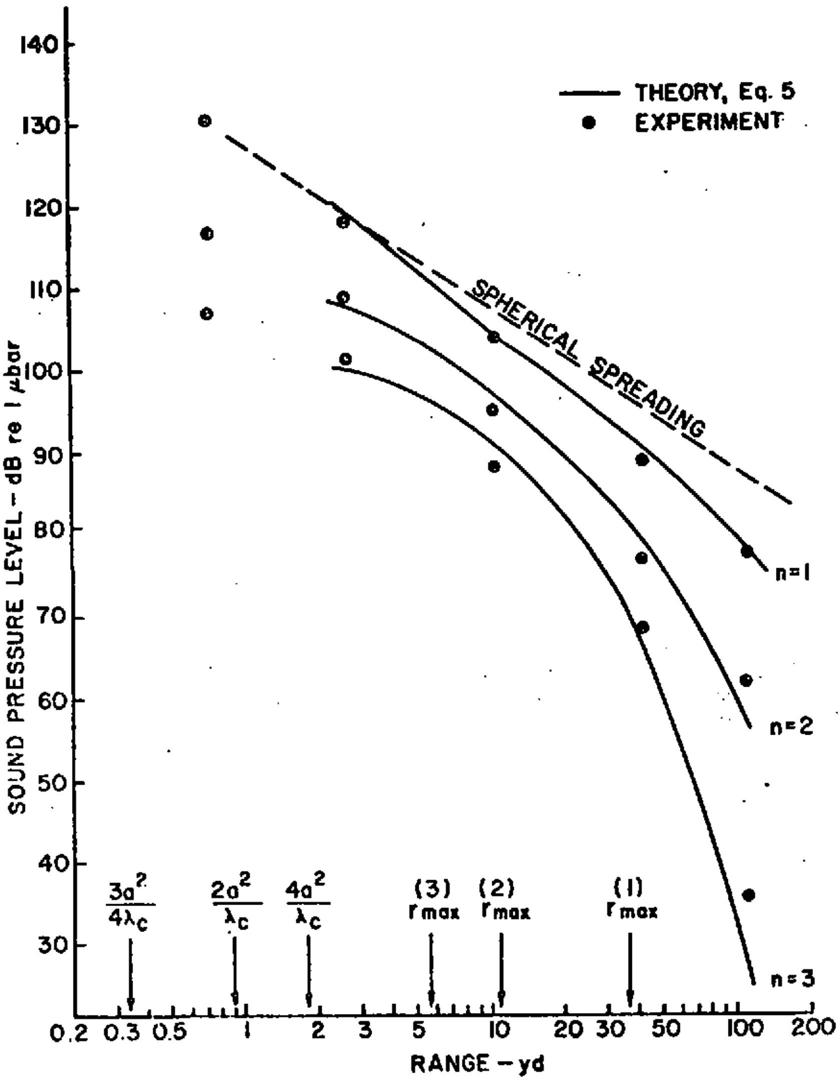


FIGURE 3
PROPAGATION CURVE
HARMONICS IN A SPHERICAL WAVE OF FINITE AMPLITUDE

ARL - UT
AS-71-59
TGM - R
5 - 26 - 7

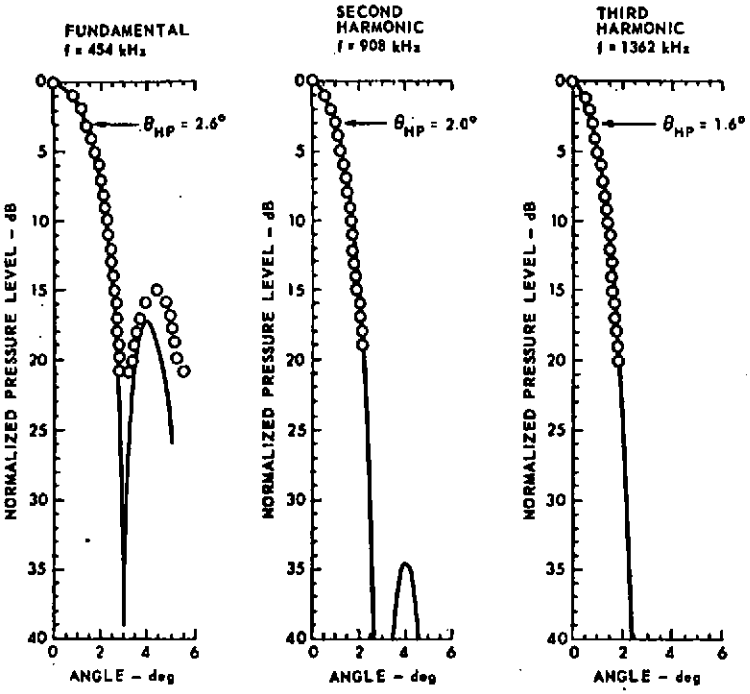


FIGURE 4
BEAM PATTERN DATA FOR WEAK WAVES
OOO EXPERIMENT —— THEORY
FUNDAMENTAL SOURCE LEVEL = 110 dB re 1 μbar at 1 yd
PATTERNS MEASURED AT R = 117 yd

ARL - UT
AS-71-342-S
TGM - DR
3-22-71

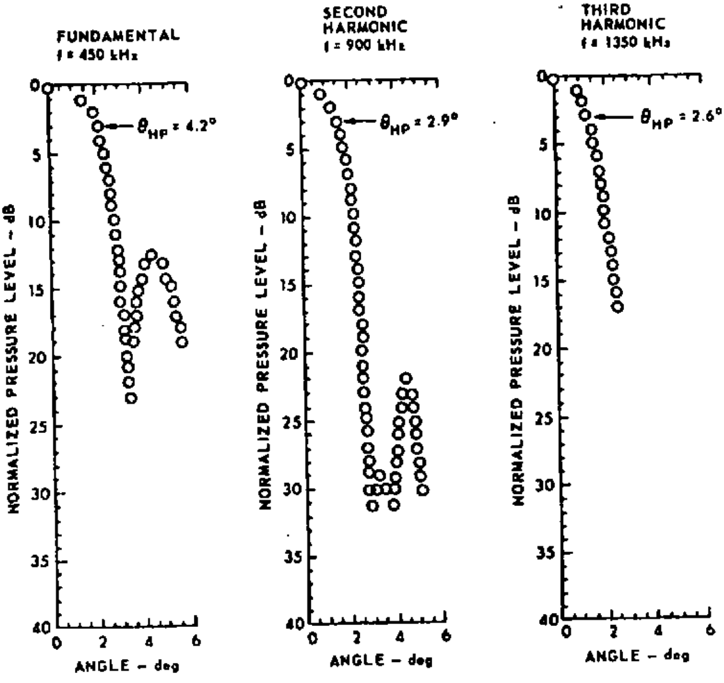


FIGURE 5
BEAM PATTERN DATA FOR STRONG WAVES
OOO EXPERIMENT —— THEORY
FUNDAMENTAL SOURCE LEVEL = 127 dB re 1 μbar at 1 yd
PATTERNS MEASURED AT R = 112 yd

ARL - UT
AS-71-343-S
TGM - DR
3 - 22 - 71