FESHBACH RESONANCE AND EXPLOSIVE INSTABILITY OF MAGNETOELASTIC WAVES

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Periodic modulation of a nonlinear material parameter is an efficient mechanism of the explosive instability in acoustics. In contrast to the conventional absolute parametric instability having the exponential development scenario, the explosive instability manifests as a singularity in the wave amplitude at some finite propagation time. Our case study concerns the explosive instability of magnetoelasic Lamb waves in a magnetic material subject to the electromagnetic modulation of the third order elastic modulus. The instability occurs due to the effect of the three-phonon coupling by means of electromagnetic field. In a general many-body problem, in particular in the problem of ultra cold atoms, it is known that particles interaction is enhanced by the Feshbach resonance. In solid-state acoustics, the resonance of Feshbach type corresponds to the nonlinear interaction of traveling elastic waves with electromagnetic pumping via an acoustic mode of discrete spectrum. This mode under the appropriate resonant conditions provides a positive feedback of the traveling waves on the pumping. The effect is predicted in an antiferromagnetic material with the "easy plane" type magnetic anisotropy. In this material, the magnetoelastic interaction induces the giant acoustic nonlinearity that can be controlled by the external magnetic field. We analyze the threshold conditions for the explosive instability and numerically simulate the explosive scenario of wave amplification over the threshold. It is shown that the explosive scenario can occur with a very low level of the incident Lamb wave amplitude comparable to the spontaneous acoustic noise. The considered mechanism of nonlinearity modulation is extendable onto systems of different physical nature and potentially applicable in acousto-electronics, electro- and hydrodynamics and in microsystems designing.
Keywords: antiferromagnetic materials, explosive instability, three-phonon interaction

1. Introduction

Instabilities in dynamic systems attract the attention of researchers due significant their positive and negative impacts. On one hand, development of instabilities efficiently generates noise, vibrations, etc., finally resulting in fatigue, wear, damage of components and structures, as well as poor or unacceptable for the human society exploitation conditions. On the other hand, instabilities help convert external pumping energy into mechanical energy or produce giant amplification of useful signals. These factors motivate our interest to establishing the conditions of the huge amplitude growth effect and to finding exemplar systems in which these conditions can be theoretically predicted and experimentally observed.

There are at least two confirmed types of behaviour characterized by a theoretically infinite amplitude growth: exponential and explosive. The exponential instability appears when a linear parameter such as stiffness in oscillators or sound velocity in acoustics is efficiently modulated by another physical process. This effect is usually called parametric amplification and is typical for a wide range of situations ranging from the classical pendulum with a variable string length to stimulated processes in laser physics [1], light scattering [2,3], acoustics [4], etc.

Our interest here is to another type of growing instabilities having the explosive behaviour. In this case, an external process modulates not the linear parameter but the quadratic nonlinear coefficient. The difference between the "usual" parametric instability having the exponential character and the explosive effect of nonlinearity modulation can be understood using the Hamiltonian formalism. The classical Hamiltonian contains terms with two amplitudes in the former case and with three amplitudes in the latter case. Application of the appropriate resonance conditions produces terms containing a combination of two or three complex conjugate amplitudes, respectively. In a quantum counterpart of such an interaction between two or three phonons, two or three creation operators appear. The presence of the third creation operator explains an additional contribution to the amplification process and results in an explosive amplitude growth when theoretically infinite values are obtained at a finite moment of time, as it is for the mathematical singularity.

The shear mode that considerably alters the system's behaviour pays a contribution that mathematically resembles the Feshbach resonance [5,6] supporting the bound molecular states in the quantum system of ultracold atoms. This is the reason for calling the effect the Feshbach-type resonance.

2. Equations for the explosive instability in antiferromagnetics

The objective of this section is to derive equations describing the explosive instability in a particular system where the three-phonon interaction appears. The system represents an antiferromagnetic plate in which a Lamb wave propagates in the presence of pumping by means of an alternative magnetic field. In addition, a shear standing wave is to be generated. In this situation, another Lamb wave with the opposite propagation direction is spontaneously excited. The three phonons necessary for the explosive instability generation are coming from the two Lamb waves and from the shear resonance mode. The magnetic pumping action modulates the quadratic nonlinear parameter and actually provides energy for the explosive amplitude growth.

As a model medium we choose an antiferromagnetic crystal with the magnetic anisotropy of the "easy plane" type belonging to symmetry group D_{3d}^6 (e.g. α-Fe_2O_3 or $FeBO_3$). The crystal has a shape of a plate cut in the basal plane normal to the crystallographic axis $C_3 \parallel z$ (see Fig. 1). We suppose that the plate is placed in a constant magnetic field $\vec{H}$ parallel to y-axis and in a transversal RF magnetic field $\vec{h}_p(t)$ parallel to the binary axis $U_2 \parallel x$ (see Fig. 1). The instability effect is produced by the interaction of the fundamental shear mode with the in-plane displacements parallel to the binary axis x and two asymmetric Lamb waves with polarization normal to the plane and with the wave vectors $\pm k$ parallel and antiparallel to the x-axis.

It is possible to show [7] that the potential energy density in the material has the form:

$$F = 2C_{44}u_{xz}^2 + \Psi_p h_p(t) u_{xz}^3, \tag{1}$$

where ρ is density of the crystal, C_{44} is the shear elastic modulus and Ψ_p is the amplitude of interaction caused by modulation of the nonlinear elastic parameter $C_{555}(\vec{H})$:

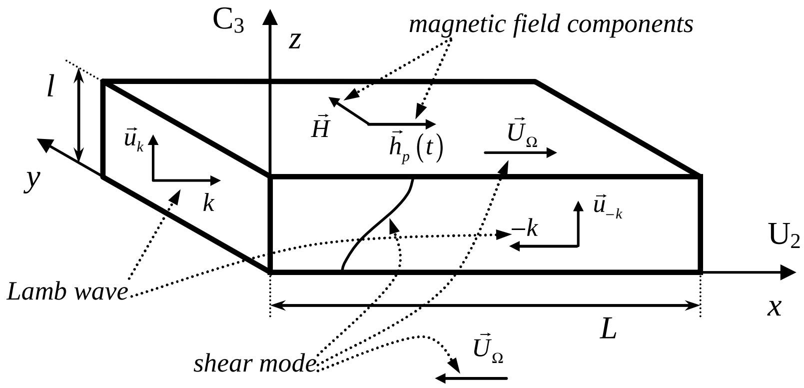

Figure 1: System's geometry. Wave displacements $\vec{u}_k$ and $\vec{u}_{-k}$ for the Lamb waves with wave vectors $\vec{k}$ and $-\vec{k}$ are shown as well as wave displacement $\vec{U}_\Omega$ for the shear mode. Magnetic fields $\vec{H}$ and $\vec{h}_p(t)$ are also plotted.

$$\Psi_p = \frac{1}{3}\frac{\partial}{\partial H_x} C_{555}\left(\vec{H}\right), \tag{2}$$

An explicit expression for Ψ_p applicable to the antiferromagnet of D_{3d}^6 symmetry with the easy plane type magnetic anisotropy in transversal alternative magnetic field is derived in [8]. In the particular case when the only nonzero strain component is U_{xz}, Ψ_p equals to

$$\Psi_p = -16C_{44}\zeta^4 \frac{1}{\varepsilon}\frac{H+H_D}{\left(\omega_{s0}/\gamma\right)^2}\Xi, \tag{3}$$

where

$$\Xi = 1 - \frac{H_D H_E H_{ms}}{2\left(H+H_D\right)\left(\omega_{s0}/\gamma\right)^2}, \tag{4}$$

In Eqs. (3)-(4), $\varepsilon=2B_{14}/C_{44}$ is the spontaneous magnetostrictive strain, B_{14} is a magnetoelastic constant, H_E, H_D and H_{ms} are exchange, Dzyaloshinsky and magnetoelastic effective fields, respectively, ω_{s0} is the frequency of antiferromagnetic resonance, γ is the magneto-mechanical ratio, ζ is the magnetoelastic coupling coefficient. The details of this derivation can be found in [7].

In Eq. (1), the pumping magnetic field that modulates the quadratic nonlinearity coefficient is chosen as

$$h_p(t) = h_0 e^{i\omega_p t} + c.c., \tag{5}$$

where ω_p is pumping frequency, h_0 is the magnetic field amplitude.

In order to show that the explosive instability appears in such a system, it is sufficient to assume that the displacement field has the following structure:

$$u_x = (De^{i\Omega t} + D^* e^{-i\Omega t})\cos\left(\frac{\pi}{l}z\right), \tag{6}$$

$$u_z = \left(Ae^{-ikx} + Be^{ikx}\right)e^{i\omega_k t}\sin\left(\frac{\pi}{l}z\right) + c.c.. \tag{7}$$

Here the contribution u_x corresponds to the shear resonance mode with the frequency Ω and amplitude D, while u_z-component describes the Lamb waves with the correspondent wave number k

and frequency ω_k. The Lamb waves have approximately vertical displacement since they are considered in the short-wave approximation in order to make use of the fact that wave interactions enhance when the wavelength decreases. Amplitude *A* of the forward wave is coming from the excitation signal while the backward wave of the amplitude *B* is not deliberately excited but appears spontaneously as it will be demonstrated. In Eqs. (6)-(7), *l* is the plate thickness.

The equations of motion corresponding to the potential energy density Eq.(1) have the form:

$$\rho\frac{\partial^2 u_x}{\partial t^2}=C_{44}\frac{\partial^2 u_x}{\partial z^2}+\frac{3}{2}\Psi_p h_p(t)u_{xz}\left(\frac{\partial^2 u_x}{\partial z^2}+\frac{\partial^2 u_z}{\partial z\partial x}\right), \tag{8}$$

$$\rho\frac{\partial^2 u_z}{\partial t^2}=C_{44}\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_x}{\partial x\partial z}\right)+\frac{3}{2}\Psi_p h_p(t)u_{xz}\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_x}{\partial z\partial x}\right). \tag{9}$$

Equations for amplitudes are obtained from Esq. (8)-(9) in the following way. First, Eq (8) is multiplied by $\cos\left(\pi z/l\right)$, Eq. (9) is multiplied by $\sin\left(\pi z/l\right)$, and both equations are integrated over the plate thickness i.e. for $0\le z\le l$. Then two resulting equations are obtained, into which the explicit forms Eq. (6)-(7) have to be substituted. The substitution produces equations for slow amplitudes *A*, *B*, and *D* together with their double derivatives. Each slow amplitude is multiplied by a rapidly evolving factor containing a combination of frequencies ω_p,ω_k, and Ω. In these equations, a number of terms can be neglected. Firstly, since amplitudes *A*, *B*, and *D* evolve slowly in comparison to fast terms with frequencies ω_k, ω_p, and Ω, their double derivatives can be omitted. Secondly, only resonant terms with

$$\omega_p-2\omega_k-\Omega\approx 0. \tag{10}$$

should be retained. Indeed, as we will show in the next section, such resonance mathematically guaranties an explosive growth of all amplitudes. Consequently, in comparison with the rapidly growing resonant terms, all other equation terms responsible for slower processes that do not have an explosive character can be neglected.

The eventual result for the slowly varying amplitudes is presented in the form of the following equations:

$$\frac{\partial A}{\partial t}+v\frac{\partial A}{\partial x}+\delta_1 A=-\frac{i}{\rho\omega_k l}\Psi_p h_0 k^2 D^* B^* e^{i\left(\omega_p-2\omega_k-\Omega\right)t}, \tag{11}$$

$$\frac{\partial B}{\partial t}-v\frac{\partial B}{\partial x}+\delta_1 B=\frac{i}{\rho\omega_k l}\Psi_p h_0 k^2 D^* A^* e^{i\left(\omega_p-2\omega_k-\Omega\right)t}, \tag{12}$$

$$\frac{\partial D}{\partial t}+\delta_2(D-D_0)=-\frac{i}{\rho\Omega l}\Psi_p h_0 k^2\frac{1}{L}\int_0^L A^* B^* e^{i\left(\omega_p-2\omega_k-\Omega\right)t}, \tag{13}$$

where damping factors δ_1 and δ_2 have been additionally introduced. Here *v* is the group velocity of the Lamb waves.

A numerical solution to Eqs. (11)-(13) is discussed in the next section.

Here it is appropriate to mention that an attempt to build up the classical Hamiltonian corresponding to Eqs. (11)-(13) will produce a term containing $h_0 e^{i\omega_p t}\left(d^*+d\right)a^*b^*+c.c.$, where *a*, *b*, and *d* are the canonical variables corresponding to amplitudes *A*, *B*, and *D*, respectively. The combination $d^*a^*b^*$ has the quantum counterpart in the form of production of three phonon creation operators. This fact indirectly explains the explosive growth effect.

3. Numerical solutions with the explosive instability

For the numerical analysis, it is convenient to rewrite Eqs. (11)-(13) in the following form:

$$\frac{\partial A}{\partial t}+\frac{\partial A}{\partial x}+\delta_1 A=-i\Phi B^*\left(D_0+D\right)^*, \tag{14}$$

$$\frac{\partial B^*}{\partial t}-\frac{\partial B^*}{\partial x}+\delta_1 B^*=i\Phi A\left(D_0+D\right), \tag{15}$$

$$\frac{\partial D}{\partial t}+\delta_2 D=-i\mu\frac{1}{L}\int_0^L dxA^*B^*. \tag{16}$$

Here time t is measured in microseconds, x and L are normalized on the group velocity v, new amplitudes A and B are obtained by adding a factor k/ε, amplitude D is multiplied by $\pi d\varepsilon/2$ (ε is the spontaneous magnetostrictive strain introduced above), detuning from resonance $\Delta\omega_p=\omega_p-2\omega_{k_0}-\Omega$ is neglected, the interaction amplitude Φ is defined as

$$\Phi=\frac{2k\varepsilon}{\pi\rho v_g}\Psi_p h_0, \tag{17}$$

and a new parameter $\mu=\Phi\Omega/\left(8\omega_k\right)$ is introduced. Further, variable D present in Eqs. (14)-(17) is an additional component of the total shear mode amplitude $D+D_0$, where parameter D_0 corresponds to a continuous excitation of the resonance mode by an external alternative force. Basically, in experiments such force is created by an additional alternative magnetic field applied at the eigenfrequency of the mode [7,9].

Equations (14)-(16) are to be completed by the boundary and initial conditions:

$$A\big|_{x=0}=A_0\left(t\right),\ A\big|_{t=0}=0, \tag{18}$$

$$B\big|_{x=L}=0,\ B\big|_{t=0}=0, \tag{19}$$

$$D\big|_{t=0}=D_0, \tag{20}$$

where $A_0(t)$ is the amplitude of an incident wave at the entrance x=0 of the active zone.

Equations (14) and (15) describe the parametric phase conjugation of travelling waves through the presence of complex conjugate amplitudes in the right-hand sides. These conjugate amplitudes contribute into Eqs. (14) and (15) together with the shear excitation D and variable Φ corresponding to the pumping magnetic field (see Eq. (17)). At the same time, Eq. (16) introduces a feedback effect into the system, when the signal (travelling Lamb waves) impacts the pumping (shear resonance). In the absence of the feedback effect, the amplitudes of Lamb waves would exponentially increase [10,11] once the threshold of parametric instability is reached. As we will show here, the addition of feedback in Eq. (16) considerably modifies the behaviour of the system. Due to the feedback, the exponential amplification scenario is followed by the explosive instability.

Accepting the following typical values of physical parameters of the problem: $\omega_k/(2\pi)$=20 MHz, $\Omega/(2\pi)$=1 MHz, acoustical quality factor of 10^3, $v=4\cdot 10^5$ cm/s, L= 4 cm, H=0.5 kOe, h_0=40 Oe, and magnetic parameters for the antiferromagnetic crystal taken from [7,12], we obtain the normalized parameters $\delta_1=6\cdot 10^{-2}$ (μs)$^{-1}$, $\delta_2=3\cdot 10^{-3}$ (μs)$^{-1}$, L=10 μs, Φ=10 (μs)$^{-1}$, $\mu=6.25\cdot 10^{-2}$ (μs)$^{-1}$ in Eqs. (14)-(16). In Fig. 2 below, t, x, and L are measured in microseconds.

In the boundary condition Eq. (18), an explicit form for $A_0(t)$ should be set. In fact, in the situation of the giant amplification considered here the exact shape of the "starter" signal is not essential. We choose a Gaussian pulse

$$A_0(t) = A_0 \exp\left(-\frac{(t-t_0)^2}{2w^2}\right), \tag{21}$$

of duration w=0.5 μs centred at t_0=2 μs. Two remaining parameters, A_0 and D_0, determining the boundary conditions Eqs. (18)-(20) are already normalized on the spontaneous magnetostrictive strain $\varepsilon \approx 10^{-5}$. Therefore $A_0=10^{-2}$ taken here as an example corresponds to a low strain of about 10^{-7}. The shear mode amplitude D_0 plays the pole of a pumping; a chosen value $D_0=5\cdot 10^{-2}$ actually means that the considered pumping amplitude is quite low (about $5\cdot 10^{-7}$) and can be increased at least by a factor of 10^1-10^2. The normalized amplitudes can reach values of order of 10^2 (physical strains about 10^{-3}); at higher strains the crystal fails.

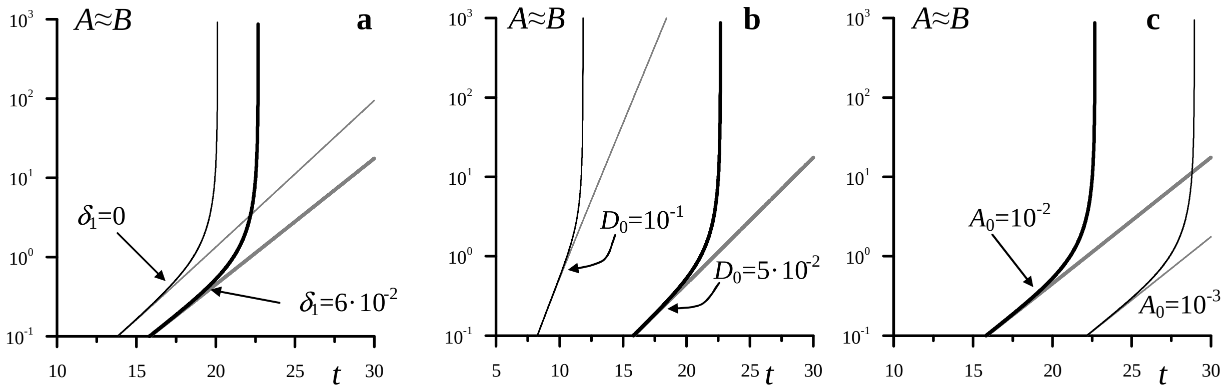

Figure 2: Time dependencies for the amplitudes $A \approx B$ at the centre of the plate i.e. at $x=L/2$ showing explosive (black curves) and exponential (gray curves) instabilities. The former case occurs in the presence of the Feshbach resonance i.e. when the additional resonant shear mode pumping is applied while the latter situation corresponds to the classical parametric interaction (no additional shear action, magnetic pumping only). The vertical axis is shown in the logarithmic scale. Sets (a)-(c) illustrate the process at different values of parameters δ_1, D_0, and A_0, respectively. The baseline curves (thick lines) are the same in all the three sets.

Figure 2 demonstrates the existence of the explosive instability at a given set of parameters (thick black line in each set (a)-(c)). At the beginning of the amplification process i.e. for 10 μs<t<20 μs in our example, the amplitudes A and B grow exponentially similarly to the case of the classical parametric interaction $\omega_p = \omega_k + \omega_{-k}$ (thick gray line), when shear resonant feedback is absent i.e. D is kept constant, $D=D_0$, instead of considering the time-dependent evolution of D according to Eq. (16). The shear pumping (Feshbach-type resonance) starts playing its role at $t \approx 22.5$ μs when a singularity develops almost instantly. The Lamb wave amplitudes immediately reach values of 10^2-10^3 in our example shown in Fig. 2 and then infinitely grow. It was also found that at large amplitudes $A \approx B$.

The three sets (a)-(c) in Fig. 2 illustrate the explosive instability dependencies on the system's parameters. In the considered example the "explosion" always occurs, but its time depends on δ_1 (set a), D_0 (set b), and A_0 (set c). If no acoustic attenuation δ_1 in the system is present, the explosion appears 2 μs earlier than in the baseline case (thick lines) but the general behaviour remains unchanged. Doubling the initial shear pumping amplitude D_0 (set b) results in a considerable enhancement of the "explosive" properties; the instability appearance time becomes twice shorter. Finally (set c), the initial Lamb wave amplitude lowered in 10 times delays the explosion development but, again, does not alter the behaviour of the system.

Generally, Fig. 2 illustrates the efficiency of the Feshbach-type resonance for enhancing the instability. Indeed, when the singularity has developed, the amplification coefficient increase due to the nonlinear feedback is theoretically infinite and, in practice, is limited by the ultimate strength of the material or high-order nonlinear effects such as pumping exhaustion, nonlinear frequency shift etc. Those effects are not considered here.

4. Conclusions

The analysis and numerical examples we present are related to systems with two-and three-phonon interactions. Two-phonon processes described here correspond to the classical parametric interaction of the kind $\omega_p = \omega_k + \omega_{-k}$, where the pumping wave of frequency ω_p exponentially amplifies signals at frequencies ω_k and ω_{-k}. In the considered case, Lamb waves of frequencies ω_k and ω_{-k} propagate in a plate made of antiferromagnetic material in which a transverse alternative magnetic field of frequency ω_p is applied. The situation changes considerably if an additional pumping channel is introduced in the form of a shear resonant mode of frequency Ω. The corresponding three-phonon process $\omega_p = \omega_k + \omega_{-k} + \Omega$ generates instabilities of much more "powerful" (explosive) type when time dependencies of signal amplitudes behave as a mathematical singularity. This offers an opportunity to convert the magnetic energy into mechanical energy in an extremely efficient manner.

An antiferromagnetic crystal excited in a way described here is only an example of a situation in which the explosive instability effect based on the three-phonon interaction is expected. The considered nonlinearity modulation mechanism is possible to extend on systems of different physical nature and to apply in acousto-electronics, electro- and hydrodynamics and in microsystems designing.

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