MACROMECHANICAL MODELLING OF ELASTIC AND VISCO-ELASTIC COSSERAT CONTINUUM AND WAVES FORMATION IN A SUCH A MEDIUM

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It has been demonstrated that the dynamics of Cosserat continuum (micropolar medium) can be simulated as a motion of two rods which elastically or visco-elastically interact with each other. The relations between elastic constants of Cosserat medium and parameters of the rods which has been initially given, have been identified. It is for the first time when visco-elastic operator has been introduced into micropolar continuum.
Keywords: Cosserat medium, solitary waves, modelling

1. Introduction

The appearance of the Cosserat continuum model [1] marked the beginning of the transition in the theory of continuous medium from the Newtonian mechanics, where the original object is the material point, to the Euler mechanics, which has a solid body as the original object. The model of Cosserat medium has a wide interest among researchers as the continuum generalization of Euler mechanics equations. Insufficient to date practical applicability of the Cosserat model is determined by the lack of reliable methods for identifying material constants and the lack of the concept of viscosity account in such a medium.

In this paper an attempt is made to find a clue how to introduce the viscoelastic operators in Cosserat model equation with the help of composite (laminate) visco-elastic rod, identical in their dispersion properties to the model of Cosserat medium.

2. Modeling visco-elastic Cosserat continuum

In the work [5] it has been shown that if plane shear wave and rotational wave propagate along the axis $x_1 = x$, they can be described with the system of equations:

$$\begin{aligned} &\rho\frac{\partial^2 v}{\partial t^2} - (\mu+\alpha)\frac{\partial^2 v}{\partial x^2} - 2\alpha\frac{\partial\psi}{\partial x} = 0,\\ &I\frac{\partial^2 \psi}{\partial t^2} - (\gamma+\varepsilon)\frac{\partial^2 \psi}{\partial t^2} - 2\alpha\frac{\partial v}{\partial x} + 4\alpha\psi = 0. \end{aligned} \qquad (1)$$

Here $v(x,t), \psi(x,t)$ - components of displacements and rotation vectors respectively; ρ - density of the medium; I - constant, which characterizes the inertial properties of the macro volume. It equals to the product of the moment of inertia of a particle of the medium about any axis passing through its center of gravity and the number of particles per unit volume; λ, μ – Lame constants; α, γ, β, ε – physical modulus of the Cosserat continuum which characterize elastic properties of the medium and satisfy the following conditions [2-4]:

$$\alpha \geq 0, \gamma+\varepsilon \geq 0, 3\beta+2\gamma \geq 0, -(\gamma+\varepsilon) \leq \gamma-\varepsilon \leq (\gamma+\varepsilon).$$

The system (8) can be reduced to one equation with respect to shear displacements:

$$2\frac{\partial^2 v}{\partial t^2}-2\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2}+\frac{I}{2\alpha}\frac{\partial^4 v}{\partial t^4}-\frac{(\mu+\alpha)I+\rho(\gamma+\varepsilon)}{2\alpha\rho}\frac{\partial^4 v}{\partial t^2\partial x^2}+\frac{(\mu+\alpha)(\gamma+\varepsilon)}{2\alpha\rho}\frac{\partial^4 v}{\partial x^4}=0, \qquad (2)$$

Waves which are described with the help of system (1) or equation (2) have the dispersion. The law of dispersion is given by the solutions to the following equation [5]:

$$\frac{\mu+\alpha}{\rho}\frac{\gamma+\varepsilon}{I}k^4+\left(\frac{4\alpha\mu}{I\rho}-\omega^2\left(\frac{\mu+\alpha}{\rho}+\frac{\gamma+\varepsilon}{I}\right)\right)k^2-\omega^2\left(\frac{4\alpha}{I}-\omega^2\right)=0. \qquad (3)$$

Consider further the composite rod, which is a set of two bars (layers) which are in contact with each other, and the force of contact interaction is assumed to be linear visco-elastic. The movement of the rods describes with the help of the system of equations:

$$\begin{aligned} E_1S_1\frac{\partial^2 u_1}{\partial x^2}&=\rho_1S_1\frac{\partial^2 u_1}{\partial t^2}+R(u_1-u_2)+R_1\left(\frac{\partial u_1}{\partial t}-\frac{\partial u_2}{\partial t}\right),\\ E_2S_2\frac{\partial^2 u_2}{\partial x^2}&=\rho_2S_2\frac{\partial^2 u_2}{\partial t^2}+R(u_2-u_1)+R_1\left(\frac{\partial u_2}{\partial t}-\frac{\partial u_1}{\partial t}\right), \end{aligned} \qquad (4)$$

where u_i – longitudinal displacements of the rods, $E_i, S_i, \rho_i (i=1,2)$ – their parameters (Young modulus, squares of transversal intersections and densities), R, R_1– coefficients of elastic and viscous interactions between the rods.

Firstly, let us consider the case when the force of contact interaction between the rods is linear elastic, i.e. $R_1=0$. Then the system (4) can be reduced to a single equation with respect to displacements one of the rods, for example u_1. For that it is enough to express the u_2 from the first equation and substitute into the second. The result is an equation in the following form:

$$\left(1+\frac{\rho_1S_1}{\rho_2S_2}\right)\frac{\partial^2 u}{\partial t^2}-\left(C_2^2+C_1^2\frac{\rho_1S_1}{\rho_2S_2}\right)\frac{\partial^2 u}{\partial x^2}+\frac{\rho_1S_1}{R}\left(\frac{\partial^4 u}{\partial t^4}-(C_1^2+C_2^2)\frac{\partial^4 u}{\partial t^2\partial x^2}+C_1^2C_2^2\frac{\partial^4 u}{\partial x^4}\right)=0. \qquad (5)$$

Here $u=u_1(x,t), C_1=\sqrt{\frac{E_1}{\rho_1}}, C_2=\sqrt{\frac{E_2}{\rho_2}}$ – velocities of longitudinal waves in the rods. The equation (5) coincides in form with the equation of the dynamics of the Cosserat medium (2). This fact gives an opportunity to simulate medium with microstructure with the help of the movement of laminated rod. Comparing the coefficients of the equations (5) and (2), we obtain a system of equations that allows to express the elastic constants of micropolar medium through the parameters of the rods:

$$\begin{gathered} 1+\frac{\rho_1S_1}{\rho_2S_2}=2;\ C_2^2+C_1^2\frac{\rho_1S_1}{\rho_2S_2}=2\frac{\mu}{\rho};\ \frac{\rho_1S_1}{R}=\frac{I}{2\alpha};\\ \frac{\rho_1S_1}{R}(C_1^2+C_2^2)=\frac{(\mu+\alpha)I+\rho(\gamma+\varepsilon)}{2\alpha\rho};\ C_1^2C_2^2\frac{\rho_1S_1}{R}=\frac{(\mu+\alpha)(\gamma+\varepsilon)}{2\alpha\rho}. \end{gathered} \qquad (6)$$

The expressions (6) can be represented in the following form:

$$\frac{\mu}{\rho}=\frac{C_2^2+C_1^2}{2},\frac{\gamma+\varepsilon}{I}=C_2^2,\frac{\mu}{\rho}=\frac{C_2^2-C_1^2}{2},\frac{I}{2\alpha}=\frac{\rho_1S_1}{R}. \qquad (7)$$

From (4) follows one more expression:

$$\frac{\beta+2\gamma}{I}=\frac{2C_2^2C_1^2}{C_2^2+C_1^2}. \qquad (8)$$

As it follows from (7), (8) the velocity of the shear wave $C_\tau=\sqrt{\frac{\mu}{\rho}}$ and velocities of rotational waves $C_\psi=\sqrt{\frac{\gamma+\varepsilon}{I}}, C_\beta=\sqrt{\frac{\beta+2\gamma}{I}}$ in medium with microstructure depend only on velocities of longitudinal waves in the rods. For a variety of composite materials the relations between the characteristic velocities of elastic waves may be different. In particular, in [6] it is shown that for the composite "aluminum fraction in the epoxy matrix" characteristic velocities related as follows: $C_\beta>C_\psi>C_\tau$. Since the rods parameters are set in advance, in order to be compliant with the Cosserat medium it is necessary that the density and cross-sectional areas are proportional $\left(\frac{\rho_1}{\rho_2} = \frac{S_1}{S_2}\right)$ and Young modulus are chosen in the way so that $\frac{E_2}{\rho_2} > \frac{E_1}{\rho_1}$.

To introduce friction in Cosserat continuum model let us assume, that the force of the contact interaction between the rods is linear visco-elastic ($R_1 \neq 0$) and the motion of the rods is described by the system of equations (4). As in the previous case, it can be reduced to one equation. Thus, adding the two equations of the system (4), we obtain the relationship in the form:

$$\rho_1 S_1 \frac{\partial^2 u_1}{\partial t^2} - E_1 S_1 \frac{\partial^2 u_1}{\partial x^2} = E_2 S_2 \frac{\partial^2 u_2}{\partial x^2} - \rho_2 S_2 \frac{\partial^2 u_2}{\partial t^2}. \tag{9}$$

From the first equation it follows:

$$R u_2 + R_1 \frac{\partial u_2}{\partial t} = \rho_1 S_1 \frac{\partial^2 u_1}{\partial t^2} - E_1 S_1 \frac{\partial^2 u_1}{\partial x^2} + R u_1 + R_1 \frac{\partial u_1}{\partial t} \tag{10}$$

and the obtained expressions (9), (10) substitute into the second equation of the system. As a result we obtain the equation with respect to $u = u_1(x,t)$ which differs from the equation (5) only by the presence of dissipative terms:

$$\frac{\partial^2 u}{\partial t^2} - \frac{C_2^2 + C_1^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\rho_1 S_1}{2R}\left(\frac{\partial^4 u}{\partial t^4} - (C_1^2 + C_2^2)\frac{\partial^4 u}{\partial t^2 \partial x^2} + C_1^2 C_2^2 \frac{\partial^4 u}{\partial x^4}\right) + \frac{R_1}{R}\left(\frac{\partial^3 u}{\partial t^3} - \frac{C_2^2 + C_1^2}{2}\frac{\partial^3 u}{\partial t \partial x^2}\right) = 0. \tag{11}$$

All other terms remain unchanged, so that (11) can be interpreted as the equation of Cosserat medium with internal friction. Here $\frac{R_1}{R}$ - the coefficient of dissipation and elastic constants of micropolar medium are linked with the parameters of the rod with the help of expressions (6) and (7). Thus, the visco-elastic operator has been introduced micropolar continuum. In the Cosserat model the dissipation is determined by terms proportional to u_{ttt} and u_{txx} that resembles the Maxwell's model of internal friction [7].

For the analysis of dispersion and dissipative properties of the waves let us change the variables in the equation (11) to the dimensionless ones: $t' = \frac{C_1^2 + C_2^2}{2}\frac{t}{r}$, $x' = \frac{x}{r}$, $u' = \frac{u}{u_0}$, where u_0 – characteristic amplitude of the wave, $r = \sqrt{\frac{C_1^2 + C_2^2}{2}\frac{\rho_1 S_1}{R}}$ – some spatial scale. As a result, the equation (11) takes the form (strokes over the dimensionless variables are omitted):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial t^2 \partial x^2} + d\frac{\partial^4 u}{\partial x^4} + \frac{1}{2}\frac{\partial^4 u}{\partial t^4} + \delta\left(\frac{\partial^3 u}{\partial t^3} - \frac{\partial^3 u}{\partial t \partial x^2}\right) = 0. \tag{12}$$

There are two dimensionless parameters in (10). One of them $d = \frac{2C_1^2 C_2^2}{(C_1^2 + C_2^2)^2}$ defines the dispersion, but the other $\delta = \frac{R_1}{\sqrt{\rho_1 S_1 R}}$ – defines dissipation. For the dispersion parameter is easy to get an estimation, if we use the Cauchy inequality of arithmetic and geometric means $\left((a+b) > 2\sqrt{ab}, a > 0, b > 0, a \neq b\right)$. It is obvious, that the dispersion parameter $d < \frac{1}{2}$, and the presence of dissipation leads to the fact that frequency and wave number of linear wave are connected through complex dispersion relation:

$$\omega^2 - k^2 + \omega^2 k^2 - dk^4 - \frac{1}{2}\omega^4 + i\delta\omega^3 - i\delta\omega k^2 = 0. \tag{13}$$

The equation (13) is a biquadratic one with respect to the wavenumber k. If we resolve it, we obtain the dependencies in the form:

$$k_{1,2} = \frac{1}{\sqrt{2d}}\left(\omega^2 - 1 - i\delta\omega \pm \sqrt{(\omega^2 - 1 - i\delta\omega)^2 - 2d\omega^4 + 4id\delta\omega^3 + 4d\omega^2}\right)^{\frac{1}{2}} \tag{14}$$

From the (21) it follows that the wavenumber is a complex $k = k' + ik''$, where $k' = Re(k), k'' = Im(k)$. This means that the wave has the constant of propagation $k'$and exponentially decays with damping coefficient k''.

On the dispersion plane (ω, k'), where k' is the real part of the complex wavenumber k, there are two dispersion branches emanating from the origin of coordinates. In this case one branch at low frequencies is close to the straight line where $\omega = k'$, but at high frequencies tends to asymptote $\omega = \sqrt{1-\sqrt{1-2d}k'}$. The second branch emanates from the origin of coordinates along the line $\omega = \frac{2\sqrt{d}}{\delta}k'$, wherein the tilt angle decreases with increasing of dissipation coefficient δ. At high frequencies this branch tends to the asymptote $\omega = \sqrt{1+\sqrt{1-2d}k'}$ which does not depend on δ.

The qualitative form of the dispersion relations $\omega(k')$ is represented in the Figure 1 when $d = 0.25, \delta = 0.1$.

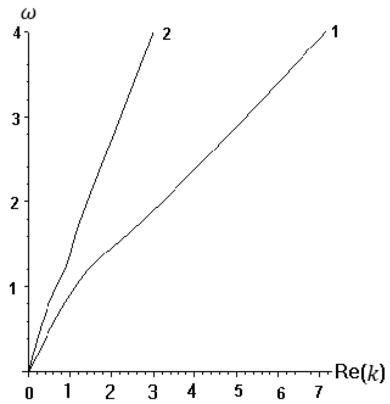

Figure 1.The dispersion characteristics of visco-elastic Cosserat medium: the dependency of the frequency on the real part of the wave number

In the Figure 2 are shown the dependencies of imaginary parts k'' of the wave number k on the frequency ω. There are two branches on the plane (k'', ω), one of them emanates from the origin of the coordinates and with the growth of the frequency tends to the horizontal asymptote $k'' = \frac{\delta(1-p^2)}{2p(2dp^2-1)}$, where $p = \frac{1}{\sqrt{1+\sqrt{1-2d}}}$. The second branch k'' emanates from the point $\omega = 0, k'' = \frac{1}{\sqrt{\delta}}$ and decreases with the increase of frequency, coming close to the horizontal asymptote $k'' = \frac{\delta(1-p_1^2)}{2p_1(2dp_1^2-1)}$, where $p_1 = \frac{1}{\sqrt{1-\sqrt{1-2d}}}$.

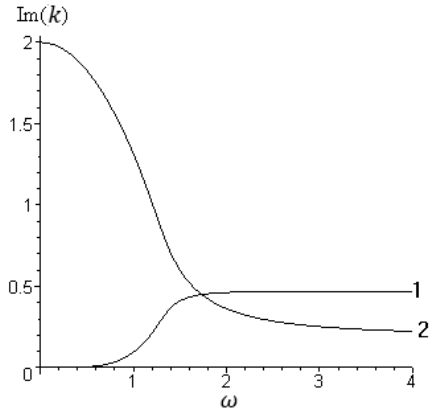

Figure 2.The dispersion characteristics of visco-elastic Cosserat medium: the frequency dependence of the imaginary part of the wavenumber

Thus, in the low frequency range the damping coefficient k'' depends on the wave frequency, but damping in the high frequency range becomes the frequency-independent, as in this case, the influence of dispersion effects become greater.

In the Figure 3 are depicted the frequency dependencies of the ratio $\frac{Re(k)}{Im(k)}$. The inequality $\frac{Re(k)}{Im(k)} > 1$ defines the frequency ranges where the process of wave propagation prevails over the process of its decaying.

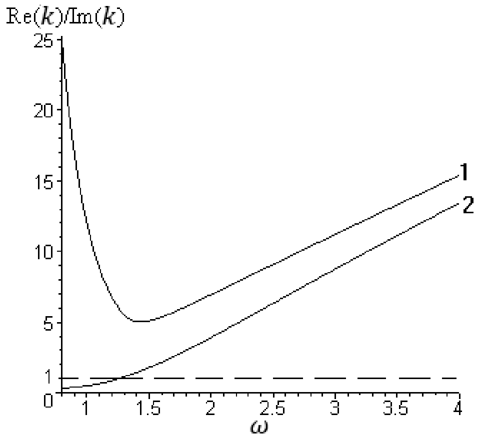

Figure 3. The dispersion characteristics of visco-elastic Cosserat medium: the frequency dependence of the ratio of real part of the wave number

3. Elastic contact of nonlinear rods

Let us consider the studying of split-designed rod, which represents the assembly of two nonlinear elastic rods (layers) which interact with each other. The dynamics of the rods can be described with the system of equations:

$$\begin{aligned} E_1S_1\left(1+\alpha_1\frac{\partial u_1}{\partial x}\right)\frac{\partial^2 u_1}{\partial x^2} &= \rho_1 S_1\frac{\partial^2 u_1}{\partial t^2}+R(u_1-u_2), \\ E_2S_2\left(1+\alpha_2\frac{\partial u_2}{\partial x}\right)\frac{\partial^2 u_2}{\partial x^2} &= \rho_1 S_1\frac{\partial^2 u_2}{\partial t^2}+R(u_2-u_1) \end{aligned} \quad (15)$$

where u_i –longitudinal displacements of the rods, $E_i, S_i, \rho_i (i = 1, 2)$ – their parameters (Young modulus, squares of transversal intersections and densities), R– coefficients of elastic interactions between the rods, $\alpha_{1,2}$ – coefficients which characterize their geometrical and physical nonlinearities.

The system (15) can be reduced to one equation. Indeed, let us introduce dimensionless variables:

$$U=\frac{u}{u_0}, y=\frac{x}{X}, \tau=\frac{t}{T}, \gamma=1+\frac{\rho_1 S_1}{\rho_2 S_2},$$

Where $D = C_2^2 + C_1^2\frac{\rho_1 S_1}{\rho_2 S_2}, X = \Lambda, T^2 = \frac{\Lambda^2\gamma}{D}, u_0$ – is a displacement, Λ – the wavelength, which satisfy the relation $\frac{u_0}{\Lambda} = 10^{-4}$, T – the period of the wave. Neglecting items with the power of the relation $\frac{u_0}{\Lambda}$ greater than three, we will get:

$$\frac{\partial^2 U}{\partial \tau^2} - \frac{\partial^2 U}{\partial y^2} + \frac{\rho_1 S_1 D}{R\gamma^2 \Lambda^2}\frac{\partial^4 U}{\partial \tau^4} - \frac{\rho_1 S_1 (C_2^2 + C_1^2)}{R\gamma \Lambda^2}\frac{\partial^4 U}{\partial y^2 \partial \tau^2} + \frac{\rho_1 S_1 C_2^2 C_1^2}{RD\Lambda^2}\frac{\partial^4 U}{\partial y^4} - \frac{\left(C_2^2 \alpha_2 + C_1^2 \alpha_1 \frac{\rho_1 S_1}{\rho_2 S_2}\right)}{R\gamma \Lambda^2}\frac{u_0}{\Lambda}\frac{\partial U}{\partial y}\frac{\partial^2 U}{\partial y^2} = 0. \quad (16)$$

Here $C_1 = \sqrt{\frac{E_1}{\rho_1}}, C_2 = \sqrt{\frac{E_2}{\rho_2}}$ – the velocities of longitudinal waves in the rods.

The solution of the equation (16) will search in the stationary waves class, i.e. in the class of the following functions $U = U(y - v\tau)$ which depend on $y - v\tau = \xi$, where $v = const$ – the velocity of the stationary wave.

The partial differential equation (16) can be reduced in this case to the equation of anharmonic oscillator with respect to longitudinal deformation $\frac{dU}{d\xi} = w$:

$$\frac{d^2 w}{d\xi^2} + aw + bw^2 = 0, \quad (17)$$

where

$$a = \frac{v^2-1}{B}, b = -\frac{1}{2}\frac{\left(C_2^2\alpha_2 + C_1^2\alpha_1\frac{\rho_1 S_1}{\rho_2 S_2}\right)}{BD}\frac{u_0}{\Lambda},$$

$$B = \frac{\rho_1 S_1 D}{R\gamma^2\Lambda^2}v^4 - \frac{\rho_1 S_1 (C_2^2 + C_1^2)}{R\gamma\Lambda^2}v^2 + \frac{\rho_1 S_1 C_2^2 C_1^2}{RD\Lambda^2}.$$

Note that the roots of the equations $B = 0$ have the following form:

$$v_1^2 = \frac{C_2^2\gamma}{D}, v_2^2 = \frac{C_1^2\gamma}{D}.$$

They particularly may satisfy the condition $\frac{C_2^2\gamma}{D} = 5 - 4\frac{C_1^2\gamma}{D}$ (for certainty, let us suppose that $C_1 > C_2$. In this case $0 < \frac{C_2^2\gamma}{D} < 1, 1 < \frac{C_1^2\gamma}{D} < \frac{5}{4}$, then $0 < v_1^2 < 1, 1 < v_2^2 < \frac{5}{4}$. The signs of the roots are the following: "$-$" inside the interval: $\frac{C_2^2\gamma}{D} < v^2 < \frac{C_1^2\gamma}{D}$ and "+" outside: $v^2 > \frac{C_1^2\gamma}{D}, v^2 < \frac{C_2^2\gamma}{D}$.

The analysis of (17) shows us that partial solutions of (16) are nonlinear stationary solitary waves (solitons).

In the first case $a < 0, b > 0$and soliton has positive polarity. The amplitude of the soliton A_c and its width Δ are described with the following relations:

$$A_c = \frac{3(v^2-1)D}{\left(C_2^2\alpha_2 + C_1^2\alpha_1\frac{\rho_1 S_1}{\rho_2 S_2}\right)\frac{u_0}{\Lambda}}, \Delta = \frac{2}{\sqrt{\frac{v^2-1}{B}}}$$

In the Figure 4 there are depicted the dependencies of the amplitude and the width of the soliton on its velocity.

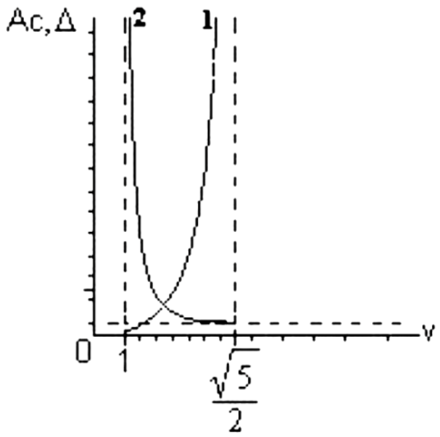

Figure 4. The dependencies of amplitude (curve 1) and width (curve 2) of the soliton of positive polarity on its velocity.

In this case with the growth of the velocity of the solitary wave its amplitude increases while width decreases. Such a behavior is typical for classical soliton [3].

For the second case $a < 0, b < 0$ and soliton has negative polarity. Its amplitude and width are described with the following expressions:

$$A_c = \frac{3(1 - v^2)D}{\left(C_2^2\alpha_2 + C_1^2\alpha_1\frac{\rho_1 S_1}{\rho_2 S_2}\right)\frac{u_0}{\Lambda}}, \Delta = \frac{2}{\sqrt{\frac{1 - v^2}{B}}}$$

The dependencies of amplitude and width of the soliton on its velocity are shown in the Figure 5

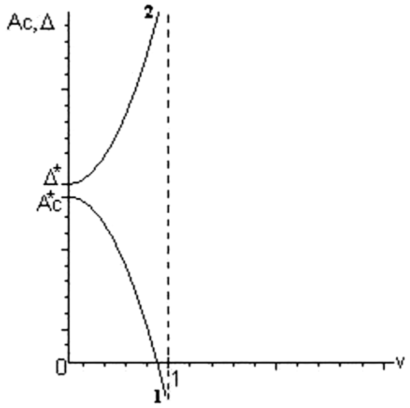

Figure 5. The dependencies of amplitude (curve 1) and width (curve 2) of the soliton of negative polarity on its velocity.

In this case with the increase of the velocity of the soliton, its amplitude and width grow simultaneously. Such a behavior is not typical for classical soliton and is abnormal.

Thus in the work it has been shown that in the split-designed nonlinear elastic rod can form localized waves (solitons) of deformations which have both positive and negative polarity.

4. Conclusions

We have proposed the approach how elastic and visco-elastic micropolar medium can be simulated with the help of laminated rod. We have introduced a visco-elastic operator into Cosserat continuum.

The comparison of dispersion curves in both cases shows that the dissipation has an impact on the dispersion properties of the wave only in the low frequency range. At the high frequency range, dissipation does not occur, since the dispersion branches at $\delta = 0$ and $\delta \neq 0$go to the same asymptote.

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REFERENCES

1. Cosserat E. et F. *Theorie des Corps Deformables*. Paris. Librairie Scientifique A. Hermann et Fils, 1909. – 226 p.
2. Nowacki W. *Theory of Asymmetric Elasticity*. Oxford, New York, Toronto et al: Pergamon Press. 1986. 383 p.
3. Eringen A.C. *Microcontinuum Field Theories. I*. Foundations and Solids. Berlin, Heidelberg, New York et al: Springer-Verlag. 1999. 325 p.
4. Eringen A.C. *Microcontinuum Field Theories. II*. Fluent Media. Berlin, Heidelberg, New York et al: Springer-Verlag. 2001. 342p.
5. Erofeyev V.I. *Wave Processes in Solids with Microstructure*. New Jersey, London, Singapore et al: World Scientific Publishing 2003. 256 p.
6. Gauthier R.D., Jahsman W.E. A *Quest for Micropolar Elastic Constants*. Arch. Mech., 1981. V.33, No 5. P.717-737.
7. Blanter M.S., Golovin I.S., Neuhauser H., Sinniug H.-R. I*nternal Friction in Metallic Materials: A Handbook*. Berlin, Heidelberg: Springer.2007. 542 p.