NUMERICAL APPROXIMATIONS FOR DETECTION- AND FALSE ALARM PROBABILITY OF A SQUARE LAW DETECTOR WITH INTEGRATOR.

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The scope of this paper is to present closed form algebraic approximations for false alarm and detection probability of a general purpose detector: A square law rectifier followed by a lowpass filter (integrator) and a binary threshold circuit.

Assuming normal, white bandpass noise and rectangular lowpass filter results in a χ^2-distribution of the rectified, smoothed noise. Hence, false alarm probability, p_f, is given by

$$p_f = Q[N\Theta | N] \tag{1}$$

where $Q[\chi^2|\nu] = \int_{\chi^2}^{\infty} t^{\frac{\nu}{2}-1} \exp[-\frac{t}{2}] dt / (2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2}))$;
N,ν = number of degrees of freedom, $N=2B/W$, B=bandpass bandwidth, W=lowpass cutoff frequency, $W \leqq B$, $\Theta = u_{th}/\eta$, η = mean (dc) voltage after the rectifier, u_{th}= threshold voltage. Both, the one- and two channel version of an envelope detector are described by $N=2$.

If signal and noise can be assumed to have the same statistical and spectral properties, distinguished only by their variance, σ_s^2 or σ_n^2 , the distribution function of the rectified and smoothed signal+noise mixture will be a scaled version of Eq.1. Hence, detection probability, p_d, can be expressed as

$$p_d = Q[N\frac{\Theta}{1+\gamma} | N] \tag{2}$$

where $\gamma = \sigma_s^2/\sigma_n^2$.

Numerical simulation of sonar operation requires repeated evaluation of Eqs. 1 and 2, or their inverse forms, $\Theta[p_f,N]$ and $\gamma[p_d,N,\Theta]$. It would be desirable, therefore, to have an invertable, closed form expression for approximating Q with negligible error at arbitrary values of all parameters.
A small step into this direction is given with the following formulas:

Chi^2-integral:

$$Q[\chi^2|N] \approx \frac{\mathrm{erfc}\left[\left(\left(\frac{\chi^2}{N}\right)^a - b\right)c\right]}{\mathrm{erfc}[-bc]}, \qquad (3)$$

where
$$a = \frac{1}{3} + .256/(((\ln N)^{1.5} - 2.4)^2 + 10);$$

$$b = 4N/(4N+1);$$

$$c = \sqrt{N}/(2a);$$

erfc = complementary error function (see approximation given below).

The maximum relative error of Eq.3 - including the effects of the erfc-approximation - is

$$\left|\frac{\Delta Q}{Q}\right| < .09 \quad \text{for} \quad 2 \leq N \leq 400,\ 0 \leq \Theta \leq 7, \text{ and } Q > 10^{-7}.$$

Inserting $\chi^2 = N\Theta$ in Eq.3 gives p_f according to Eq.1; similarily, inserting $\chi^2 = N\Theta/(1+\gamma)$ gives p_d.

Eq.3 can be easily inverted to obtain

$$\frac{\chi^2}{N} = \left(b + \frac{1}{c}\,\mathrm{erfci}\left[Q\ \mathrm{erfc}[-bc]\right]\right)^{\frac{1}{a}}, \qquad (4)$$

where erfci= inverse complementary error function (see approximation below).

Inserting $Q=p_f$ in Eq.4 signifies $\frac{\chi^2}{N} = \Theta$. Inserting $Q=p_d$ signifies $\frac{\chi^2}{N} = \Theta/(1+\gamma)$, from which γ is readily found.

ROC-curves can now be expressed in closed form (obtained from inserting Θ, given by Eq.4, into p_d, given by Eq.3)

$$p_d = E\ \mathrm{erfc}\left[bc(F-1) + F\ \mathrm{erfci}[p_f/E]\right], \qquad (5)$$

where $E = (1+\gamma)^{-a}$, and $F = \mathrm{erfc}[-bc]$.

Following Middleton (1) the "minimum detectable signal" is the signal-to-noise ratio, γ, at the detector input, required for achieving a given p_f/p_d combination. Therefore, Eq. 5 is resolved for γ:

$$\gamma = \left(\frac{bc + \mathrm{erfci}\left[p_f \ \mathrm{erfc}[-bc]\right]}{bc + \mathrm{erfci}\left[p_d \ \mathrm{erfc}[-bc]\right]} \right)^{1/a} - 1 \quad . \qquad (6)$$

<u>Special case: The envelope detector:</u>

For N=2 the above formulas take their simplemost form:

$$p_f = \exp[-\Theta] \quad ; \qquad (7)$$

$$p_d = \exp[-\Theta/(1+\gamma)] \quad ; \qquad (8)$$

$$p_d = p_f^{\,1/(1+\gamma)} \quad ; \qquad (9)$$

$$\gamma = \frac{\ln p_f}{\ln p_d} - 1 \quad . \qquad (10)$$

<u>Complementary error function:</u>

Definition

$$\mathrm{erfc}[x] = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp[-t]\, dt \quad . \qquad (11)$$

Approximation

$$\mathrm{erfc}[x] = \begin{cases} x \geqq 0: & \varphi[x] \\ x < 0: & 2 - \varphi[-x] \end{cases} \quad , \qquad (12)$$

where

$$\varphi[z] \approx \exp\left[-\sqrt{z^4 + 1.27\, z^2}\right] / \sqrt{1 + .1z + .88\, z^2} \quad .$$

The maximum relative error is

$$\left| \frac{\Delta\varphi}{\varphi} \right| < .004 \quad \text{for} \quad 0 \leqq z \leqq \infty \, ;$$

$$\left| \frac{\Delta\varphi}{\varphi} \right| < .05\, z \quad \text{for} \quad 0 \leqq z \leqq .08 \quad .$$

Inverse complementary error function:

Definition

$$\mathrm{erfci}\left[\mathrm{erfc}[x]\right] = x \quad . \tag{13}$$

Approximation

$$\mathrm{erfci}[y] = \begin{cases} 0 \leq y \leq 1 : & \psi[y] \\ 1 < y \leq 2 : & -\psi[2-y] \end{cases} \quad , \tag{14}$$

$$\text{where } \psi[z] = \sqrt{z - \ln z - 1} \,/(1.05 - .252\, z + .001 \ln z) \quad .$$

The maximum relative error is

$$\left|\frac{\Delta\psi}{\psi}\right| < .01 \text{ for } 10^{-18} \leq z \leq 1 \quad .$$

Conclusion:

A closed form algebraic approximation for the chi^2-Integral has been presented. It allows to derive simple formulas for false-alarm, detection probability and ROC-curves, and to invert these formulas. All formulas have been programmed and tested on a pocket calculator (TI SR-52). In practical application care must be taken to stay within the range of validity of N, Θ and Q, as specified with Eq.3. Outside this range the approximation errors can be much worse than the limit of 9% indicated above.

It is questionable, however, whether it would be wise to try to extent the range of validity of Θ and Q, because it would imply to rely more heavily on the validity of the far-off tails of the initially assumed Gaussian distribution of the detector input.

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PHASE FLUCTUATION OF AN UNDERWATER ACOUSTIC SIGNAL
AN APPROACH TO ESTIMATION*

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ABSTRACT

Telecommunications theory associates a characterizing function, known as the "scattering function", with the transmission medium of a general transmission channel. Under given boundary conditions, given data rate requirements, and with a knowledge of the statistical distributions of noise (WSSUS) in the time domain, it is possible to define the scattering function of the medium and use it to design the optimal signal for such a channel. The mathematical model of this function has been assessed and experiments in underwater acoustic channels have been made to verify the feasibility of such a method. However, the scattering function was expressed only in terms of "time spreading" and "frequency smear" of the transmitted monocromatic signal which, to some extent, proved inadequate for more general and more accurate evaluations. To improve the definition of the scattering function it is necessary to introduce also the space and phase parameters. This paper presents a practical approach for the estimation of the phase fluctuation of a real acoustic signal after several miles of travel through the ocean along a direct path. The work described in this paper was supported by the Italian Navy (MMI) and the Italian National Research Council (CNR).

1. GENERAL

The scattering function of an underwater transmission channel has been theoretically studied and experimentally verified [1 and 2] and in both cases certain hypotheses have been assumed in order to match the "mathematical model" with the "real physical system" which, in effect, must be considered as a "quasi-real physical system".

The scattering function is defined as a function of time spreading and frequency smear:

$$\sigma(\tau,\varphi)$$

provided by the WSSUS hypothesis (Wide Sense Stationary Uncorrelated Scattering). A more general definition of such a function, that is closer to the real physical situation, may be formulated by considering at least two further parameters of the transmission channel, i.e. the spatial configuration and the phase behaviour.

A tentative approach to introduce the spatial parameters has been developed by others [2]. The introduction of the phase parameter requires a more complex formulation that, in principle, may be represented by the following relation:

$$\sigma = \sigma(\tau, [\varphi:\phi], x)$$

where:

τ = time spreading
φ = frequency smear
ϕ = phase fluctuation
x = spatial configuration.

The frequency smear and the phase fluctuation are enclosed together in square brackets to indicate their interconnection.

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2. THE BASIC MODEL

A very general transmission channel is considered with attention mainly concentrated on the evaluation of the phase fluctuation of a real acoustic signal transmitted through an underwater transmission channel and detected by a receiver at a distance of 18 miles in an ocean depth of about 300 metres. The received signal may be considered as a signal generated by a stabilized oscillator, in reality not perfectly stable. The signal generated by such an ideal oscillator may be considered quasi-monocromatic and expressed as follows:

$$X(t) = A(t) \cdot \cos[\omega_o t + \Phi(t)]$$

where :

ω_o = fundamental angular frequency
$A(t)$ = instant amplitude
$\Phi(t)$ = instant phase.

The time dependence of A and Φ is the cause of the imperfect monocromaticy [5]. To represent a real phenomenon it is, therefore, convenient to use signal models depending upon a limited number of parameters.

3. RANDOMNESS

The signal model represents the system oscillator by means of a random function $x(t,p)$, where t is the time and p is the point of a probability space $\{P\}$; evidently, this is a statistical model. Generally, one random function $x(t)$ is said to be of the second order if $E[X^2(t)]$ is limited for any t. Most random functions are valid only in this case and, in particular, all those related to spectral decomposition.

These assumptions enable the "instant frequency" of second order to be introduced into the model, but the "instant phase" of such a system oscillator is not usually of the same type. The most evident example of phase fluctuation is that of Brownian motion.

4. STATIONARITY

The stationarity is not a property that is attainable through experience, since any physical measurement has finite duration. The frequency-phase relationship may be formulated as follows:

$$\Phi(t) = \Phi_o + \int_o^T f(t)\, dt$$

it must be noted that if $f(t)$ is stationary, it is not necessarily the same for $\Phi(t)$; this statement must, however, be accepted in a wide sense. If it is reasonable to introduce a stationary frequency $f(t)$, with a non-stationary phase $\Phi(t)$, the conclusion is that it is impossible to establish a phase-correlation function. The measurements are made in finite intervals of time T, introducing a cut frequency of the order $1/T$. It is then possible to utililize a stationary model that is observed only during the time T.

5. PHASE-FREQUENCY RELATIONSHIP

The output signal of the system oscillator is represented by

$$X(t,p) = A \cos[\omega_o t + \Phi(t,p)]$$

Whatever the statistical properties of the random function $\Phi(t,p)$, $X(t)$ is a random function of the second order [5]. The total instant phase can be deduced from $x(t)$

$$\Phi_{ist} = \omega_o t + \Phi(t)$$

where $\Phi(t)$ is the instant phase.

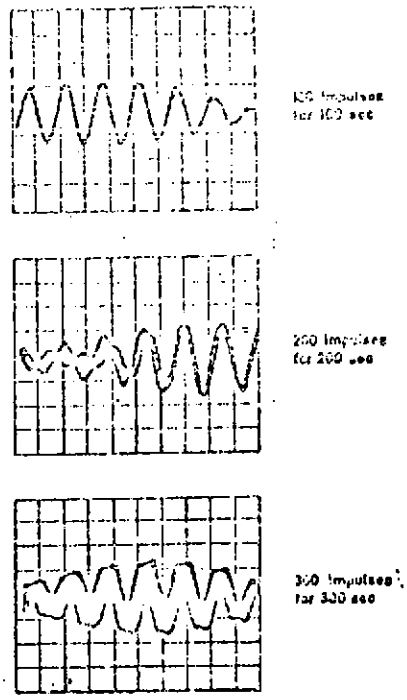

FIG. 1

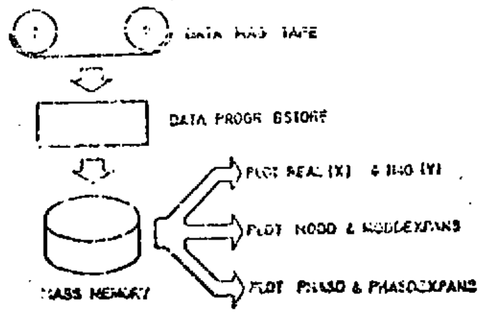

FIG. 2

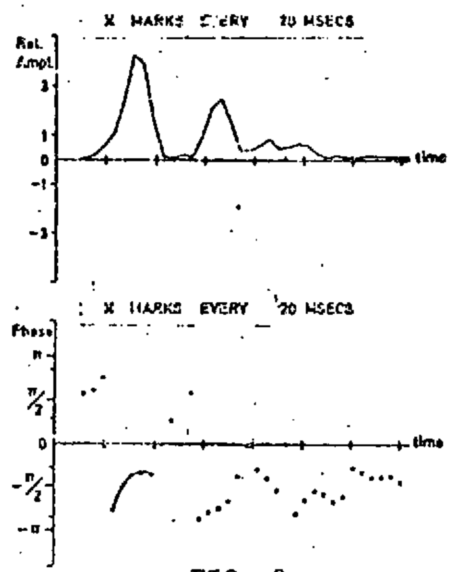

FIG. 3

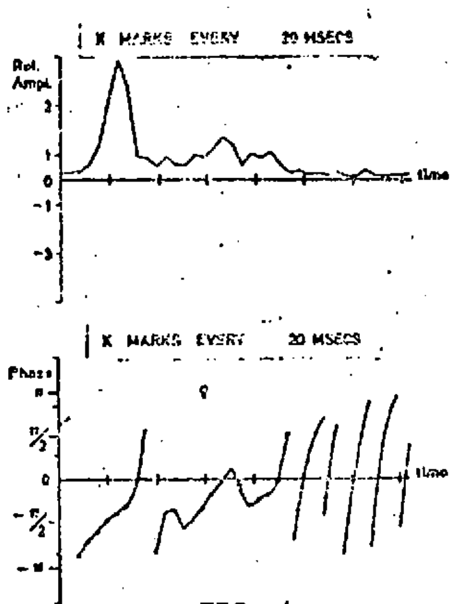

FIG. 4

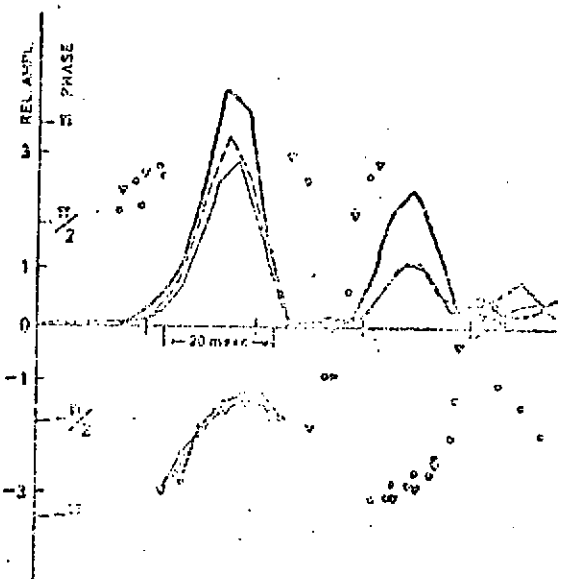

FIG. 5

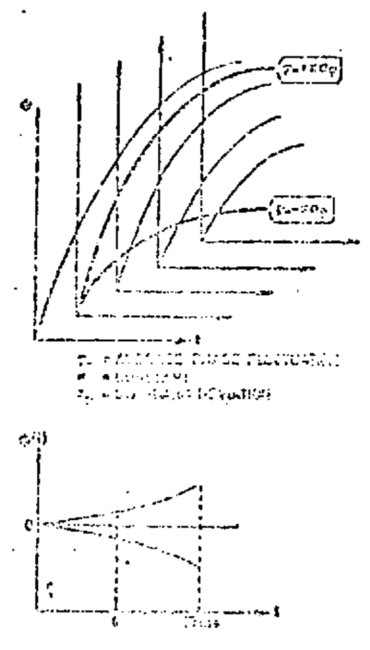

FIG. 6

For computing the phase increment we have

$$\Delta\Phi(t,\tau) = \Phi(t) - \Phi(t-\tau) \quad .$$

Similarly as in oscillators the instant frequency may be defined as a derivative of the instant phase.

$$\omega(t) = \omega_o + f(t)$$

where $f(t) = \frac{d}{dt}\Phi$

is evidently an angular frequency.

The instant frequency is not always definable or physically measurable and, as a consequence, this is also the case for phase $\Phi(t)$.

For the measurement of the instant frequency, the phase increment is taken into account and the value of the measurement is:

$$f_{\tau}(t) = \frac{1}{\tau}\Delta\Phi(t)$$

provided the instant frequency exists, we have

$$\lim_{\tau \to o} f_{\tau}(t) = f(t) \quad .$$

The computation of the phase fluctuation of a real acoustic signal is based on the properties of the phase in an instant frequency model [5].

6. EXPERIMENTAL RESULTS

The transmitted signal, composed of a series of pulses of 12 m sec length at -3dB, and of Gaussian shape with carrier frequencies of 750 Hz, 1500 Hz and 2800 Hz, was received at a distance of about 18 miles from the transmitter. The data used represent those relevant to a direct path, namely the first arrival. The detected signal was sampled and decomposed in quadrature in sin and cos components and recorded on a digital magnetic tape [3].

The instant phase can then be immediately computed:

$$\Phi_{iSt} = \text{arctang}\,\frac{X(t)}{Y(t)} \quad .$$

The instant phase is given if the preliminary filtering is relatively narrow banded; the phase value in the wider band is an average phase.

The practical estimation of the phase fluctuation of such a signal [Fig 1] may be approached by the following procedure:

(i) Data reduction [Fig 2].

(ii) Selection of first arrival (direct path) [Figs 3a and 4a]..

(iii) Computation of instant phase for real direct path.

(iv) Statistics over a large number of direct-path pulses at various frequencies and different periods, represented by oceanographic parameters [Figs 5 and 6].

7. CONCLUSIVE CONSIDERATIONS

The evaluation outlined in this paper represents only the starting point of a complex process initiated some years ago. Fluctuation is an up-to-date argument as shown in the recent JASA Meeting at Miami Beach, U.S.A., in December 1977. Direct-path evaluation of the phase fluctuation has as a consequence the volume scattering estimate. From the very first observation it appears that the phase fluctuation for a direct path is poor and this result is in agreement with recent results achieved by other scientists.

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