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PROPAGATION OF ELASTIC WAVES IN CONSTRAINED ROTATING MECHANICAL SYSTEMS

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1. INTRODUCTION

Khulief and Shabana [1], proposed a method for the dynamic analysis of flexible multibody systems with intermittent motion. The jump discontinuities in the system velocities and reaction forces are predicted using the generalized impulse momentum equations that involve the coefficient of restitution and the Jacobian matrix of the kinematic constraints. The validity of using the generalized impulse momentum equations in the impact analysis of constrained deformable bodies has been examined both theoretically and analytically by Rismantab-Sany and Shabana [2]. The convergence of the series solutions obtained by solving the generalized impulse momentum equations was used to prove the validity of using these equations in the impact analysis of constrained deformable bodies. Yigit, et al. [3], examined experimentally the validity of using the impulse momentum equations. Their experimental results were in a good agreement with the numerical results obtained using the impulse momentum equations. They concluded that the generalized impulse momentum equations can be used with confidence to study impact problems in constrained deformable bodies.

The objective of this investigation is to examine the effect of the finite rotational displacement on the velocity of propagation in constrained elastic systems. The system differential equations of motion are developed using the principle of virtual work in dynamics. The jump discontinuities in the system generalized velocities as the result of impact are predicted using the impulse momentum equations that involve the coefficient of restitution. The impact-induced wave motion is analyzed using Fourier method. The series solution obtained in this paper is written as the sum of transient and steady state solutions. The use of the procedure described in this paper is demonstrated using a radially rotating rod which is subjected to an axial impact.

2. DYNAMIC EQUATIONS

In this investigation the radially rotating rod shown in Fig. 1 is used to examine the effect of the finite rotation on the propagation of the impact-induced longitudinal elastic waves. Using the principle of virtual work in dynamics and assuming the angular velocity of the rod is specified and constant, the differential equations of motion of the rod i can be written as [4]

$$\mathbf{M}^i_{ff}\ddot{\mathbf{q}}^i_f + (\mathbf{K}^i_{ff} - \omega^2\mathbf{M}^i_{ff})\mathbf{q}^i_f = \omega^2\bar{\mathbf{I}}^i_0 \qquad (1)$$

where ω is the angular velocity of the rod and $\mathbf{M}^i_{ff}$ and $\mathbf{K}^i_{ff}$ are, respectively, the mass and stiffness matrices of the rod which can be evaluated using the assumed shape function. The matrices $\mathbf{M}^i_{ff}$ and $\mathbf{K}^i_{ff}$ and the vector $\bar{\mathbf{I}}^i_0$ are given by

$$\mathbf{M}^i_{ff} = \frac{m^i}{2}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n\times n} \tag{2a}$$

$$\mathbf{K}^i_{ff} = \frac{Ea\pi^2}{8\,l}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 9 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (2n-1)^2 \end{bmatrix}_{n\times n}, \qquad \bar{\mathbf{I}}^i_0 = \frac{4m^i l}{\pi^2}\begin{bmatrix} 1 \\ \frac{-1}{9} \\ \frac{1}{25} \\ \vdots \\ \frac{(-1)^{n+1}}{(2n-1)^2} \end{bmatrix} \tag{2b}$$

where m^i, a, l, and E are the mass, cross sectional area, length, and the modulus of elasticity of the rod, and n is the total number of vibration modes.

3. IMPACT ALGEBRAIC EQUATIONS

3.1 Impulse Momentum Equations

Figure 1 shows a mass m^j that moves along the axis of the rod with a constant velocity V^j. This mass is assumed to impact the free end of the rod axially at time $t = 0$. If the geometry of impacting surfaces and friction between the two impacting bodies are not considered, the generalized impulse momentum equations of this system is given by [1]

$$\begin{bmatrix} \mathbf{M} & \mathbf{L_q} \\ \mathbf{L_q}^T & 0 \end{bmatrix}\begin{bmatrix} \Delta\dot{\mathbf{q}} \\ H \end{bmatrix} = \begin{bmatrix} 0 \\ \nu \end{bmatrix} \tag{3}$$

where $\mathbf{M}$ is the system mass matrix which is defined as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^i_{ff} & 0 \\ 0 & m^j \end{bmatrix} \tag{4}$$

in which $\mathbf{L_q}$ is the change of the penetration in the direction of the common normal at the point of impact with respect to system generalized coordinates $\mathbf{q}$, $\Delta\dot{\mathbf{q}}$ is the jump in the velocity vector, H is the generalized impulse, and

$$\nu = -(1+e)V^j \tag{5}$$

in which e is the coefficient of restitution.

By solving Eq. 3, the jump discontinuities in the velocity vector can be written as

$$\Delta\dot{\mathbf{q}} = \begin{bmatrix} (\Delta\dot{q}^i_f)_k \\ \Delta V^j \end{bmatrix} = H\begin{bmatrix} \frac{2}{m^i}(-1)^{k+1} \\ -\frac{1}{m^j} \end{bmatrix}, \qquad k = 1, 2, \dots, n \tag{6}$$

where the generalized impulse H is given by

$$H = \frac{-\nu}{\mu + 2n} m^i \tag{7}$$

in which μ is the mass ratio defined as

$$\mu = \frac{m^i}{m^j} \tag{8}$$

3.2 Solution of the Differential Equations
By using Eq. 6, the initial conditions for the differential equation of Eq. 1 as the result of the axial impact are defined as

$$[q_j^i(0)]_k = 0, \qquad [\dot{q}_j^i(0)]_k = (\Delta \dot{q}_j^i)_k = \frac{2H}{m^i}(-1)^{k+1}, \quad k = 1, 2, \ldots, n \tag{9}$$

By using these initial conditions, Eq. 1 can be solved for the elastic coordinates. This leads to

$$(q_j^i)_k = A_k \cos\beta_k t + B_k \sin\beta_k t - A_k \tag{10}$$

where

$$A_k = \frac{2\eta_k^2(-1)^k}{lP_k^2(1-\eta_k^2)} \quad , \qquad B_k = \frac{(\Delta \dot{q}_j^i)_k}{\beta_k} \tag{11}$$

in which the wave number P_k, the circular frequency β_k, and the dimensionless rotation-wave number η_k are defined, respectively, as

$$P_k = \frac{(2k-1)\pi}{2\,l} \quad , \quad \eta_k = \frac{\omega}{c_0 P_k} \quad , \quad \beta_k = P_k c_0 \sqrt{1-\eta_k^2} \tag{12}$$

in which c_0 is the velocity of the wave propagation without dispersion and is defined as

$$c_0 = \sqrt{\frac{E}{\rho}} \tag{13}$$

in which ρ is the mass density of the rod.

4. WAVE MOTION IN THE ELASTIC SYSTEM

In this section the propagation of the impact-induced waves is analyzed using Fourier method, wherein the wave motion is represented by using the mode superposition technique. The longitudinal displacement of an arbitrary point on the rod after impact can be obtained as

$$u_j^i(x,t) = \sum_{k=1}^{n} (\sin P_k x)(q_j^i)_k \tag{14}$$

Substituting Eq. 10 into Eq. 14, the longitudinal displacement of an arbitrary point on the rod can be expressed as the sum of two functions as

$$u_j^i(x,t) = v^i(x,t) + w^i(x) \tag{15}$$

where v^i and w^i are called, respectively, the transient and steady state solutions and are defined as

$$v^i(x,t) = \frac{1}{2}\sum_{k=1}^{n} X_k \left[\cos(P_k x - \beta_k t - \phi_k) - \cos(P_k x + \beta_k t + \phi_k)\right] \tag{16a}$$

$$w^i(x) = -\sum_{k=1}^{n} A_k \sin P_k x \tag{16b}$$

in which the amplitude X_k and the phase angle ϕ_k are, respectively, defined as

$$X_k = \sqrt{A_k^2 + B_k^2} \quad , \qquad \phi_k = tan^{-1}\frac{A_k}{B_k} \tag{17}$$

Observe that the steady state solution $w^i(x)$ at a given point x is constant and does not depend on time. While the transient response $v^i(x,t)$ describes a wave motion. Furthermore, the steady state solution w^i is equal to zero if the angular velocity is equal to zero, while the transient response v^i depends on the angular velocity as well as the impact conditions.

5. PHASE VELOCITIES OF THE WAVE MOTION

In perfectly elastic structural systems, the wave motion can be represented using Fourier's method as the sum of infinite number of harmonic waves. The phase velocities of these harmonic waves are equal and equal to the group velocity of the wave motion. The medium in this case is said to be nondispersive. In this section the effect of the finite rotation on the phase velocities of the impact-induced waves is examined.

It is clear that the kth term in the transient response of Eq. 16a can be written as

$$\begin{aligned} v_k^i &= \frac{X_k}{2}\left[\cos(P_k x - \beta_k t - \phi_k) - \cos(P_k x + \beta_k t + \phi_k)\right] \\ &= f_1(x - c_k t) + f_2(x + c_k t) \end{aligned} \tag{18}$$

in which the phase velocity c_k of the kth term is defined using Eq. 18 as

$$c_k = \frac{\beta_k}{P_k} \tag{19}$$

Substituting Eq. 12 into Eq. 19, one may define the dimensionless phase velocity as

$$(c_k)_d = \frac{c_k}{c_0} = \sqrt{(1 - \eta_k^2)} \tag{20}$$

One may observe, in view of Eq. 20, that if ω or equivalently η_k is equal to zero, c_k is equal to c_0 and all the harmonic waves have the same phase velocity defined by Eq. 13. Observe that as the result of the finite rotation, different harmonic waves have different phase velocities. Clearly, the finite rotation of the rod has more significant effect on the low frequency harmonic waves as compared to the high frequency harmonic waves. Figure 2 shows the effect of the angular velocity of the rod on the velocity of the harmonic waves. The results presented in this investigation are obtained for the case in which the length of the deformable rod is assumed to be 3.6 m, and its circular cross section has a diameter of 0.0185 m. The rod is assumed to be made of steel with modulus of elasticity equal to $2 \times 10^{11} N/m^2$ and mass density equal to 7870 kg/m^3. Observe from Eq. 20 and the results presented in Fig. 2 that as the mode number k increases, the wave length λ_k decreases and the effect of the angular velocity ω on the phase velocity decreases. It is also clear, from Eq. 20, that the phase velocities of the harmonic waves are independent of the impact conditions and the coefficient of restitution. They depend only on the material of the rod as well as the finite rotation.

6. TRANSIENT WAVE MOTION

In this section the effect of the angular velocity of the rod as well as the impact conditions such as the velocity of the impacting mass V^j, the mass ratio μ, and the coefficient of restitution e on the transient wave motion is examined.

5.1 Angular velocity

Figure 3 shows the effect of the finite rotation on the transient wave motion as described by the function v_k^i in Eq. 18. Clearly, the finite rotation has a more significant effect on the low frequency modes of vibration as compared to the high frequency modes.

5.2 Impact conditions

While the impact conditions do not have any effect on the phase velocities of the harmonic waves, these conditions have an effect on the transient wave motion v^i. It is also clear that increasing the velocity of the impacting mass leads to an increase in the absolute value of the amplitude of the wave motion. It is clear that increasing the mass m^j and the coefficient of restitution e lead to an increase in the amplitude of the elastic waves.

7. TOTAL DEFORMATION

The total axial deformation at a point on the rod is the sum of the transient impact-induced wave motion and the steady state displacement resulting from the finite rotation of the rod. It may be misleading to assume based on the discussion presented in the preceding sections that the increase in the angular velocity reduces the vibration. It is important, therefore, to emphasize at this point that increasing the angular velocity does not imply a decrease in the total longitudinal deflection of the rod. Even though, for a constant angular velocity, the steady state response $w^i(x)$ given by Eq. 16b is time independent, $w^i(x)$ has a significant effect on the total longitudinal deflection of the rod. Clearly, this term is equal to zero, when the angular velocity ω is equal to zero. The total longitudinal deformation at an arbitrary point on the rod is therefore the sum of the harmonic wave motions plus this constant term which is mainly due to the angular velocity of the rod. The effect of the angular velocity ω on the total axial deformation is shown in Fig. 4. It is clear from Fig. 4 that increasing the angular velocity of the rod increases the longitudinal deflection of any point on the rod.

8. SUMMARY AND CONCLUSIONS

In this investigation the effect of the finite rotation on the propagation of elastic impact-induced waves in constrained deformable bodies that undergo large displacement is examined. Using the solution of the generalized impulse momentum equations, the solution of system differential equations is expressed as the sum of the wave motion and the steady state solution. It is shown in this paper that dispersion occurs as the result of the finite rotation. Consequently, the resulting harmonic waves travel with different phase velocities that depend on the finite rotation. It is also shown that the finite rotation has more significant effect on the phase velocity of the low frequency harmonic waves as compared to the high frequency harmonic waves. Even though in the analysis presented in this investigation, only the case of axial impact is considered, similar procedure that utilizes the generalized impulse momentum equations can be used for the analysis of transverse waves in constrained elastic systems.

9. ACKNOWLEDGEMENT

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10. REFERENCES

[1] Y KHULIEF & A SHABANA, 'Dynamic Analysis of Constrained System of Rigid and Flexible Bodies With Intermittent Motion', ASME Journal of Mechanisms, Transmissions, and Automation in Design, Vol. 108, No. 1, p38, 1986.

[2] J RISMANTAB-SANY & A SHABANA, 'On the Use of the Momentum Balance in the Impact Analysis of Constrained Elastic Systems', ASME Journal of Vibration and Acoustics, Vol. 112, p119, 1990.

[3] A S YIGIT, A G Ulsoy, & R A SCOTT, 'Dynamics of A Radially Rotating Beam With Impact, Parts I and II', ASME Journal of Vibration and Acoustics, Vol. 112, p65, 1990.

[4] A SHABANA, Dynamics of Multibody Systems, Wiley, New York, 1989.

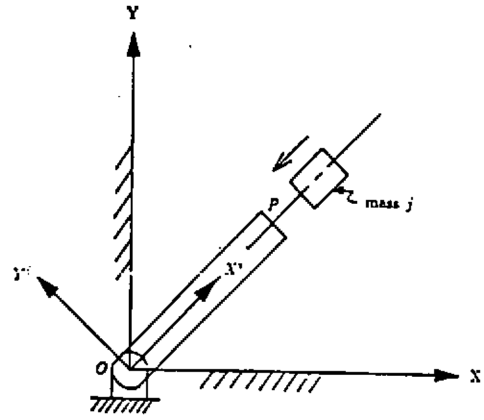

Fig. 1 Coordinates of the flexible beam.

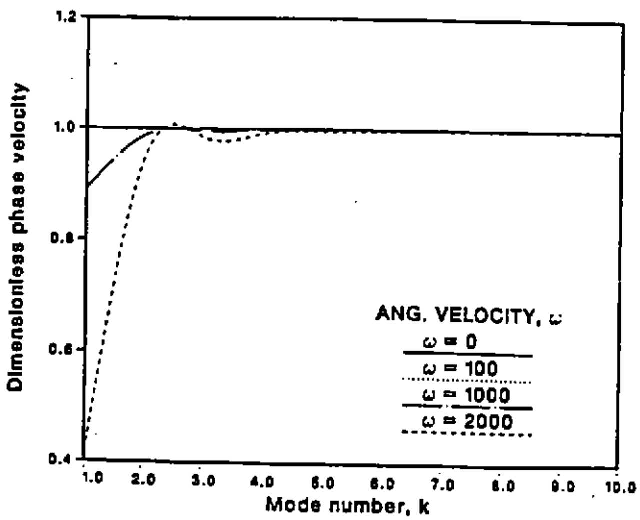

Fig. 2 The effect of the angular velocity ω on the dimensionless phase velocity.

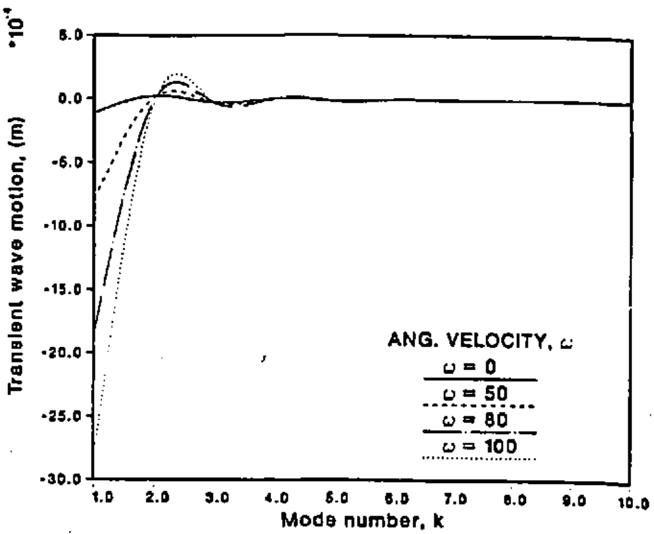

Fig. 3 The effect of the angular velocity ω on the transient wave motion at the midpoint ($e = 0.005$, $\mu = 1.4$, $t = 0.0004$ *sec*).

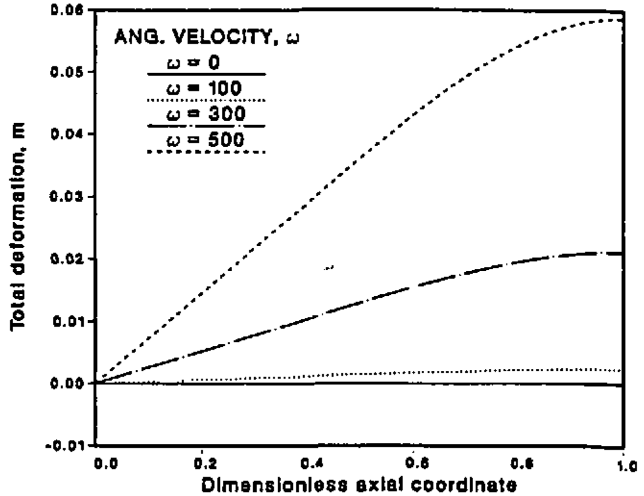

Fig. 4 The effect of the angular velocity ω on the total deformation ($e = 0.005$, $\mu = 1.4$, $n = 100$, and $t = 0.0004$ *sec*).

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