24th INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
23–27 July 2017, London

NONLINEAR ACOUSTICS ON CURVILINEAR SPACETIME

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Most of the theories and derivations of equations for nonlinear acoustics are based on the flat spacetime or Minkowski spacetime. The flat spacetime is an approximation of the curvilinear spacetime. These include the derivations of the KZK equation, Burgers equation, and Westervelt equation. This paper is an introduction to nonlinear acoustics on the curvilinear spacetime. Nonlinear acoustics is the study of finite amplitude waves and involves with strong sound field. Hence it will be a suitable candidate for the curvilinear spacetime theoretical platform. The field of nonlinear acoustics is a very broad field including acoustic radiation force which is intense sound, finite amplitude wave equations, sonolumineasence, turbulence, nonlinear theory of elasticity, nonlinear theory of piezoelectricity, phonon-phonon interaction, electron-phonon interaction. In this paper, only some of the subfields of nonlinear acoustics are covered such as acoustic radition force, elastography, acoustic force of levitation, nonlinear piezoelectricity, and finite amplitude wave equation. It is shown that there is no need for the Lighthill analogy between the Navier-Stokes equation of fluid mechanics and the acoustic equation of motion as fluid mechanics is the root of acoustics and the acoustic equation of motion is only a linearized form of the Navier-Stokes equation. Here a new finite wave equation based on the curvilinear spacetime is derived. It is more accurate than the usual KZK equation of nonlinear acoustics as KZK equation is only an approximate equation based on the nonrelativistic Navier-Stokes equation.

Keywords: curvilinear spacetime, relativistic theory

1. Introduction

The theories of acoustics and the derivation of equations of acoustics such as the Helmholtz wave equation, KZK equation, Burgers equation, Westervelt equation, Christoffel equation for crystal acoustics are all based on the flat spacetime or Minkowski spacetime. This limits to sound propagation only on a 2D plane instead on a 3D space. Also in so doing the equations will come with the special theory of relativity. In the extension of the calculations to the curvilinear spacetime, general relativity will be involved. This will also include the gravitational force. Also energy and momentum will affect the curvature of the curvilinear spacetime. Sofar the acoustical cloaking is the first example of sound propagation on a curvilinear spacetime. Curvilinear spacetime will be a new theoretical platform for the derivation of acoustic equations. There are numerous calculations in acoustics based on curvilinear coordinates. But these are only for the description of the geometrical structure of the objects under consideration. Also in acoustical cloaking only one aspect of the general theory of relativity is concerned . That is only the curvilinear coordinates aspect of the problem is considered. The property of gravitational force is not considered. It is of interest to mention that in his original paper on the foundation of general relativity[1] towards the end of the paper, Einstein mentioned about the extension from flat spacetime to curvilinear spacetime on two areas of physics, that is electrodynamics which involved with Maxwell's equations and hydrodynamics which is related to acoustics. This shows that it is time to extend the calculations and the equations of nonlinear acoustics to the curvilinear spacetime.

Areas in nonlinear acoustics to be treated on a curvilinear spacetime will include acoustic radiation force(ARF) which is applied to acoustofluidics,elastography and acoustic force of levitation., finite amplitude wave equations such as KZK equation, Burgers equation and Westervelt equation, nonlinear elasticity, nonlinear piezoelectricity, electron-phonon interaction, phonon-phonon interaction, turbulence and sonoluminescence.

Nonlinear acoustics on the curvilinear spacetime is a deductive approach. The methodological advantage lies in the fact that a test of a linear theory obtained as a special case of a more general theory provides a definite test of the general theory itself. On the other hand, if the linear theory is developed separately then a test of it will provide no futher information as to the validity of its generalization since these generalizations are not uniquely determined by the linear theory itself.

2. Relativistic acoustic radiation force

Acoustic radiation force(ARF) is an intense sound.The derivation of the ARF is from the Navier Stokes equation of hydrodynamics.The usual treatment is based on the flat spacetime or Minkowski spacetime and the gravitational force is neglected.The flat spacetime is an approximation of curvilinear spacetime.

The procedure for the derivation of the relativistic acoustic radiation force(ARF) is to derive the acoustic pressure and the particle velocity by solving the relativistic momentum equation and the relativistic continuity equation and substitute them in the expression for the acoustic potential. The. ARF is then given by the gradient of this acoustic potential. It is also given by the integration of this stress over the whole volume of the system In the curvilinear spacetime, the stress tensor is given by

$$T^{\mu\nu} = (\mathrm{e} + \mathrm{p})v^{\mu}v^{\nu} + \mathrm{p}g^{\mu\nu} \quad (1)$$

where e = total energy density, p= acoustic pressure, v= particle velocity , and g = gravitational force.

The relativistic ARF is then given by

$$\mathrm{F} = \int T^{\mu\nu}\mathrm{dS} \quad (2)$$

where S = total surface area of the system.

When applying to a spherical particle, $\mathrm{V}=\frac{4}{3}\Pi r^3$, and $\mathrm{S} = 4\Pi r^2$,$\mathrm{dS} = 8\Pi r\mathrm{d}r$.
The details on the relativistic stress tensor is given in section below.

An application of the ARF is to acoustofluidics. Here one is concerned with acoustic radiation force(ARF) on small particles. The scope will cover acoustic tweezer, drug delivery, acoustic streaming etc. When an ultrasound field is imposed on a fluid containing a suspension of particles, the latter will be affected by the acoustic radiation force (ARF) arising from the scattering of the acoustic waves on the particle. The particle motion resulting from the ARF is denoted acoustophoresis and plays a key role in on chip microparticle handling

The Einstein tensor is a measure of the curvature of spacetime. Mass is merely a form of energy and as such one denotes the stress energy tensor $T_{\mu\nu}$, containing all of the information of the energy of a system. Thus these two tensors must be in balance, which is represented in the Einstein field equations(EFE):

$$G_{\mu\nu} = \frac{8\Pi G}{c^2} T_{\mu\nu} \quad (3)$$

where G = gravitational constant, c= velocity of light.

The EFE represent a system of ten nonlinear differential equations. Due to the complexity of these equations, few analytical solutions exist.

$\nabla_\nu G^{\mu\nu}$=0 and apply this to (3),

$$\nabla_\mu T^{\mu\nu} = 0 \qquad (4)$$

This is an important equation in fluid dynamics. It encapsulates the idea of energy and momentum conservation.

Different systems will have different stress energy tensors. Often, a lot of the problems of viscosity and other effects can be neglected compared with pressure or other more dominant effects. Here one will consider the perfect fluid stress energy tensor. In the rest frame of the observer this can be written as

$$T^{\mu\nu} = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \qquad (5)$$

or in a more general frame

$$\boldsymbol{T}^{\mu\nu} = (e + p)\boldsymbol{v}^{\mu}\boldsymbol{v}^{\nu} + p\boldsymbol{g}^{\mu\nu} \qquad (6)$$

The relativistic Euler equation can be written as

$$(e+p)v^{\mu}\nabla_\mu v_\nu = - \nabla_\nu p - v_\nu v^{\mu}\nabla_\mu p \qquad (7)$$

From (7), the spatial components will be given as:

$$(e+p)\,\frac{Dv}{D\tau} = -\nabla p - v\frac{Dp}{D\tau} \qquad \text{Momentum equation} \qquad (8)$$

$$v^{\mu}\nabla_\mu e = -(e+p)\,\nabla_\mu v^{\mu} \qquad \text{Continuity equation} \qquad (9)$$

where $\frac{D}{D\tau} = v^{\mu}\nabla\mu$

3. Elastography on curvilinear spacetime

Acoustic radiation force impulse(ARFI) imaging is a form of ultrasound elastography used in medicine, particularly for the diagnosis and monitoring of cancers. ARFI uses acoustic radiation force(ARF) to generate images of the mechanical properties of soft tissue.

Acoustic radiation force is a phenomenon associated with the propagation of acoustic waves in attenuating media. Attenuation includes both scattering and absorption of the acoustic wave. Attenuation is a frequency dependent phenomenon, and in soft tissues it is dominated by absorption. With increasing acoustic frequencies, the tissue does not respond fast enough to the transitions between positive and negative pressures, thus its motion becomes out of phase with the acoustic wave, and energy is deposited into the tissue. This energy results in a momentum transfer in the direction of wave propagation and tissue heating. The momentum transfer generates a force that causes displacement of the tissue, and the time scale of this response is much slower than that of the ultrasonic wave propagation. This displacement (typically a few micrometres), which is typically detected by computing the correlation of ultrasonic RF signal, can be used to derive additional information about the tissue beyond what is normally provided in an ultrasonic image. The magnitude, location, spatial extent, and duration of acoustic radiation force(ARF) can be controlled to interrogate the mechanical properties of the tissue.

In linear, isotropic, elastic solids, the speed of shear wave propagation$(c)_T$ is related to shear modulus (μ) and density (ρ) by

$$c_T = \sqrt{\frac{\mu}{\rho}} \qquad (10)$$

Equation (10) provides a relationship between shear wave speed and shear modulus; however, there are two significant challenges to using this relationship to characterize the modulus of soft tissue: (1) generating shear waves within tissues *in vivo*, and (2) reconstructing from measured displacement fields.

Generating shear waves within tissues can be accomplished by coupling external mechanical sources through the skin into the organ of interest or generating the shear wave within tissues using acoustic radiation force. The FibroScan® system (EchoSens, Paris, France) uses an external vibrator to generate shear waves in tissue and has successfully quantified differences in liver stiffness as correlated with fibrosis stage . Similar approaches of external shear wave excitation have also been used in MR-based elastography techniques . While these findings are promising, such setups can be challenged in their ability to couple enough energy through the skin and subcutaneous fat to generate adequate shear wave displacements within organs such as the liver, especially in obese patients. External mechanical excitation sources can also be limited by the ribs when trying to reach more superior and lateral regions of the liver.

Some of these challenges can be overcome with the use of focused acoustic radiation force excitations where mechanical excitation occurs along the acoustic wave propagation path and within the focal region of the acoustic beam. These radiation force excitations generate shear waves directly in the tissue of interest. Acoustic radiation force is applied to absorbing and/or reflecting materials in the propagation path of an acoustic wave. This phenomenon is caused by a transfer of momentum from the acoustic wave to the propagation medium. The spatial distribution of the radiation force field (i.e., the region of excitation, or ROE) is determined by both the acoustic excitation parameters and the tissue properties. In soft tissues, where the majority of attenuation results from absorption , the following equation can be used to determine radiation force magnitude

$$F = W_{absorbed}/c = 2\alpha I/c \qquad (11)$$

where F [dyn (1000 cm)$^{-3}$], or [kg s^{-2} cm^{-2}], is acoustic radiation force (in the form of a body force), $W_{absorbed}$ [W (100 cm)$^{-3}$] is the power absorbed by the medium at a given spatial location, c[m s^{-1}] is the speed of sound in the medium, α [cm^{-1}] is the absorption coefficient of the medium, and I [W cm^{-2}] is the temporal average intensity at a given spatial location. The spatial extent of the ROE varies with focal characteristics and tissue attenuation; however, it is always distributed within the geometric shadow of the active transmit aperture .

Eqn(11) is for non-relativistic case. ARF is an intense sound and the relativistic form of the ARF given by eqns.(1) and (2) should be used.

4. Relativistic acoustic radiation force and the acoustic force of levitation

Acoustic levitation (or acoustophoresis) is a method for suspending matter in a medium by using acoustic radiation pressure from intense sound waves in the medium.

Acoustic levitation is usually used for containerless processing which has become more important of late due to the small size and resistance of microchips and other such things in industry. Containerless processing may also be used for applications requiring very-high-purity materials or chemical reactions too rigorous to happen in a container. This method is harder to control than other methods of containerless processing such as electromagnetic levitation but has the advantage of being able to levitate nonconducting materials.

By 2013, acoustic levitation had progressed from motionless levitation to controllably moving hovering objects, an ability useful in the pharmaceutical and electronics industries. A prototype device involved a chessboard-like array of square acoustic emitters that move an object from one square to another by slowly lowering the sound intensity emitted from one square while increasing the sound intensity from the other, allowing the object to travel virtually "downhill".

Acoustic levitation has been performed before, but with very limited control of the object's movement The breakthrough now is that researchers are able to move their acoustically levitated objects up and down, as well as side to side. The levitation effect is created by standing waves — static (standing) waves that are held in place by a reflector that bounces the same wave back upon itself, causing interference. In short, if one has enough power, one can levitate just about anything with acoustic waves — including a human. The power required would be immense, though, and at this point it isn't clear if a human would even survive the acoustic forces. Furthermore, before one starts dreaming of a portable levitation device , the power requirements would probably be well beyond the capabilities of today's lithium-ion battery packs. Levitating a water drop requires around 160 dB.

An experimental setup has been built up[4] to measure the levitation force produced by the presented disc levitation system. An aluminum plate with the same diameter as the radiation plate is positioned in opposite to the radiator. This sound-reflecting plate is mounted on a vertical linear stage through a load cell for being able to measure the vertical force acting on the reflector directly. Using the linear stage, the reflector may be positioned freely between the contact position and a distance of about 40 mm above the radiator. A laser interferometer is installed to measure the exact vertical position of the reflector.

A common compact disc (CD) is chosen as the object to be levitated. It has the same diameter as the vibrating plate, with a thickness of 1.3 mm and a mass of 16 g. A stable levitation state is observed when the input power reaches about 30 W (see Fig. 8). The CD then rests without any instable vertical motion above the flexural plate. Maximum vibration amplitude of the excitation system occurs at the center of the flexural plate and is about 25 μm at 19 kHz for this level of power (measured using a laser vibrometer). It is worth mentioning that the CD in this arrangement rests at a position slightly higher than half a wavelength (above the peak of the levitation force), where the levitation force equals the gravity force of the CD. This is different compared to common radiator-reflector-type systems, in which small particles are levitated at positions slightly below the pressure nodes of the standing wave. Stable levitation could not be achieved at one wavelength or higher positions with the proposed setup due to the quickly dropped levitation force.

Su Zhao and Joig Wallaschek [2] has derived the Rayleigh radiation pressure from the vibrating circular plate as:

$$p_{ra} = Z(r)^2 \frac{V_0^2}{4(sinh^2\propto' L + sin^2 kL)} \rho_0 \quad (14)$$

where V_0 = surface vibration amplitude, ρ_0 = ambient density, L = distance between object to be levitated and the radiator, and ∝' = increased absorption coefficient of finite amplitude wave.

Eqn.(14) is the acoustic radiation force(ARF). This will be the force of levitation if it can balance the gravitational force. Thus gravitational force plays a key position in acoustic levitation. Thus the accurate calculation of the gravitational force will enable the adjustment of eqn(14) 's parameters or the experimental parameters to enable the largest size or the heaviest object that can be levitated.

An analysis of Einstein's theory of the gravitational field and Einstein field equations of gravitation will be done for the calculation of the accurate gravitational force for this acoustic levitation system in a future paper.

5. Nonlinear Piezoelectricity

The standard piezoelectricity theory is a linear theory. Therefore it allows the use of the principle of superposition which implies among other things that there should be no change in the resonant frequency of a piezoelectric resonator when subjected to either mechanical stress, electrical stress or large amplitude s of vibration. It is therefore clear that the linear theory must be generalised in some way to account for them. The usual method is for proceeding from the particular to the general; from the special, linear case to the more general, nonlinear case. But it has become clear in recent years that in many cases it is advantages to reverse this procedure, that is, to proceed from the general to the particular, and to regard the case of a material with linear constitutive relations as a special case of a more general class of materials. A particular methodological advantage of the deductive approach lies in the fact that a test of a linear theory obtained as a special case of a more general theory provides a definite test of the general theory itself. On the other hand, if the linear theory is developed separately, then a test of it will provide no further information as to the validity of its generalizations, since these generalizations are not uniquely determined by the linear theory itself.

The current problems encountered in the design of quartz crystal resonators require for their solution a generalization of the linear theory of piezoelectricity Such a generalization can either be ad hoc,or based on a serious attempt to provide a general theory which contains the linear theory as a special case. Such a theory must be a relativistic theory of electromechanical interactions in materials. As an illustration, one will start with the action principles for elastic solids and fluids in the absence of electromagnetic fields. The requirement that the field equations for a physical system be obtainable from an action principle results in a system of exactly N equations for N unknowns. The Einstein equations .for the gravitational field potentials, viz. the components of the metric tensor on the spacetime manifold, can be derived from an action principle with a scalar density Lagrangian function given by $(-g)^{1/2}$R, were R is the scalar curvature of the spacetime manifold. The action leads to the field equations

$$G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik} = 0 \quad (15)$$

where G_{ik} is the Einstein tensor and R_{ik} the Ricci tensor and $g_{ik}$the components of the fundamental tensor. (15) are the field equations valid in matter and field free space-time.

In order to obtain the field equations in the presence of matter or electromagnetic fields, the action principle must be modified by the addition to the Lagrangian of a term describing the matter or fields present. The fundamental assumption made here is that the action principle in the general case shall have the form

$$\delta A = 0 \qquad (16)$$

where the action A is given by

$$A = \int\{-(-g)^{1/2}\frac{R}{2\kappa} + L\}\, d^4v \qquad (17)$$

where κ is the gravitational constant, L is a scalar density function describing the matter and fields present and the integral is taken over a fixed region of the spacetime manifold. The variations of the dynamical variables considered in (16) are such that they vanish on the boundary of the region of integration. They may also, be subject to constraints implied by the existence of geometric laws which have not been satisfied identically by means of a potential representation. From (16) and (17) follow the modified gravitational field equations

$$(-g)^{1/2}G_{ik} + \kappa T_{ik} = 0 \qquad (18)$$

where T_{ik} is a symmetric tensor density given by

$$T^{ik} = 2\,\frac{\delta L}{\delta g_{ik}} \qquad (19)$$

$\delta L/\delta g_{ik}$ is the Lagrange derivative of L with respect to the g_{ik}. T^{ik} is the stress=-energy –momentum tensor of the matter and fields represented by the term L in the Lagrangian density. Besides the gravitational field equations, (18), (16) implies field equations for the other variables occurring in L. The identities required by the principle of general covariance are in this case the identities following from the Bianchi identities

$$\nabla_k G^{ik} = 0 \qquad (20)$$

Viz. they are $\nabla_k T^{ik} = 0$ (21)

∇_k represents covariant differentiation with respect to x^k based on the Christoffel symbols defined in terms of the metric tensor.

The stress-energy-momentum tensor of an elastic solid can be calculated according to the prescription (19). The subsequent works will be derivation of the action principle for the electromagnetic field in free space, the action principle for a system of interacting fields and materials will be the topics of a future paper.

6. Discovery of a new finite amplitude wave equation based on curvilinear spacetime

The usual finite amplitude wave equation is the KZK equation. But this is an approximate equation based on the flat spacetime. It is of interest to point out that acoustics actually originates from fluid mechanics and hydrodynamics. Hence the acoustic equation of motion is the linearized form of the Navier-Stokes equation. In this sense, there is no need of the Lighthill analogy which states that the acoustic equation of motion is an analogus from of the Navier-Stokes equation. However, these are all based on the flat spacetime. The Navier-Stokes equation gives rise to the KZK equation as the second order approximation based on the flat spacetime. The curvilinear spacetime with the relativistic Navier-Stokes equation will give rise to a new finite amplitude wave equation with the KZK equation as an approximation.

The procedure of the derivation of the linear acoustic equation of motion from the Navier-Stokes equation will be followed to derive the new finite amplitude wave equation.

The momentum equation of the nonrelativistic Navier-Stokes equation can be given as:

$$\rho\frac{\partial v}{\partial t} = -\nabla p \qquad (22)$$

where ρ= mass density, v =particle velocity, p= acoustic pressure.

For nonrelativivistic case, one uses the Hooke's law for the stress-strain relation by writing p = stress = $\mu\frac{\partial \xi}{\partial x}$.

$$\rho\frac{\partial^2}{\partial t^2}\boldsymbol{\xi} = -\mu\frac{\partial^2}{\partial x^2}\boldsymbol{\xi} \qquad (23)$$

or
$$\frac{\partial^2}{\partial t^2}\boldsymbol{\xi} = -c^2\frac{\partial^2}{\partial x^2}\boldsymbol{\xi} \qquad (24)$$

where $c = \sqrt{\frac{\mu}{\rho}}$ = sound velocity.

Thus the nonrelativistic Navier-Stokes equation gives rise to the linear acoustic equation of motion with the linear Hooke's law in place.

To derive the relativistic finite amplitude wave equation, one uses the relativistic momentum equation part of the relativistic Navier=Stokes equation:

$$(e+p)\frac{Dv}{D\tau} = -\nabla p - v\frac{Dp}{D\tau} \qquad (25)$$

where e= total enery density .

Next one replaces p in (25) by the relativistic stress tensor:

$$T^{\mu\nu} = (e+p)v^{\mu}v^{\nu} + pg^{\mu\nu} \qquad (26)$$

Then
$$e\frac{Dv}{D\alpha} + (e+p)v^{\mu}v^{\nu}\frac{Dv}{D\tau} + pg^{\mu\nu}\frac{Dv}{D\tau} = -\nabla\{(e+p)v^{\mu}v^{\nu} + pg^{\mu\nu}\} - v\frac{D}{D\tau}\{(e+p)v^{\mu}v^{\nu} + pg^{\mu\nu}\} \qquad (27)$$

Eqn(27) is the new finite amplitude wave equation. This is more accurate than the usual KZK equation which is derived from the nonrelativistic Navier-Stokes equation of the flat spacetime.

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