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A NOVEL SPECTRAL DYNAMIC STIFFNESS METHOD FOR EFFICIENT AND EXACT ELASTODYNAMIC ANALYSIS WITHIN THE WHOLE FREQUENCY RANGE

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An efficient analytical spectral dynamic stiffness (SDS) method for exact modal analysis of elastodynamic problems is presented in this paper. The general solution satisfying the governing differential equation exactly is first derived by applying the proposed modified Fourier series. Then the SDS matrix for an element is formulated symbolically using the exact general solution. The SDS matrices are assembled directly in a similar way to that of the finite element method, demonstrating the method's capability to model complex structures. Any arbitrary boundary conditions are represented accurately in the form of the modified Fourier series. The formulation is applicable to both dynamic response and wave propagation analyses; and the method is applied to elastodynamic problems with simple as well as complex geometries. When applied in modal analysis, the Wittrick-Williams algorithm is used as the solution technique where the mode count problem (J0) of a fully-clamped element is resolved. The proposed method gives exact solution with remarkable computational efficiency, covering low, medium and high frequency ranges. All results from the theory in this paper are accurate up to the last figures quoted to serve as benchmarks. This new method offers an idea tool for parametric and optimization studies of structures, especially in the vibro-acoustic analysis within mid- to high-frequency ranges.

Keywords: Spectral dynamic stiffness method, modal analysis, whole frequency range, elastodynamic analysis, Wittrick-Williams algorithm

1. Introduction

Dynamic analysis is a major consideration during the design, construction and operational phases of structures such as high-speed trains, auto-mobiles, ships, submarines, buildings, bridges, circuit boards, amongst many others. The dynamic behaviour of such structures can be generally classified into categories within high, medium and high frequency ranges. Different types of available methods are limited to different frequency ranges. In general, dynamic analysis to analyse them is performed by the well-developed finite element method (FEM). The FEM is no-doubt versatile to handle complex geometries, but it is mostly confined to dynamic analysis within relatively low frequency range. One of the reasons for this is due to the approximate shape functions adopted in the analysis. The dynamic characteristics of structures in the medium to high frequency ranges are without doubt crucial and becoming increasingly important. For example, structures must be designed to sustain complex loading conditions (e.g., impact and blast) which may have both low and high frequency contents.

The vibroacoustic behaviour in the medium to high frequency ranges covers the most hearing sensitive range (500Hz-20kHz) for human being, thus the control of noise arising from structures in automotive, aeronautics, marine and electronic industries becomes very important. Furthermore, the structural health monitoring aims to perform non-destructive evaluation through the integrated actuators and sensors, which essentially relies on efficient algorithms to solve related inverse problems within medium to high frequency ranges. Unfortunately, existing modelling techniques such as the conventional FEM are somehow inadequate and not fit for the purpose. Therefore, one should resort to alternative methods. The dynamic analysis within the high frequency range is generally performed using energy-based method such as the statistical energy analysis (SEA) [1] and dynamic energy analysis (DEA) [2], which are based on a number of assumptions. However, these assumptions restrict their applications only to the high frequency range, and the application scope of the SEA and DEA is often questionable. some parameters are crucial to the SEA such as the coupling loss factors which are not always easy to estimate. Other alternative methods include the wave based method (WBM), spectral finite element method (SFEM) and wave finite element method (WFEM). The WBM has achieved great success [3], but it is usually confined to steady-state dynamic problems in the mid-frequency domain. The SFEM [4] is used in time domain response analysis but it is restricted to prismatic plates with simple boundary conditions. The WFEM [5] on the other hand, when applied within medium to high frequency ranges, seems only applicable to infinite periodic 2D structures for which the periodicity simplifies the problem significantly into a quasi-1D problem.

No satisfactory method seems to be available to cover the entire frequency range for plate structures. One exception is of course, the dynamic stiffness method (DSM) which encompasses both modal [6–8] and response [9] analyses. The method gives exact dynamic solutions within all frequency ranges of interest, which can be used as benchmark solutions for other methods. However, the DSM is seriously restricted to prismatic plate assemblies with the assumption that two opposite sides of each plate component must be simply supported [6–9], which prevents the DSM applications in a wider context. In order to remove the above restrictions and at the same time remain the merits (exactness and super efficiency) of the DSM, this paper will introduce a newly developed method called the spectral dynamic stiffness method (SDSM) [10–15] which accounts for both prismatic and non-prismatic plate assemblies with any arbitrary classical [10–13] and non-classical [14, 15] boundary conditions. The SDSM has been applied to modal analysis to give exact solutions within low, medium and high frequency ranges with remarkable computational efficiency. For instance, the SDSM exhibits two orders of magnitudes of improvement in computational efficiency over the conventional FEM. This superiority is much more pronounced at higher frequencies. The elegance and uncompromising accuracy of the SDSM provide a much wider appeal than the classical DSM.

2. Spectral dynamic stiffness formulation

The SDSM [10–15] combines the spectral (S) method and the classical dynamic stiffness method (DSM). One of the key points in the SDSM lies in adopting two sets of modified Fourier series. The adopted modified Fourier series for any arbitrary displacement or force boundary condition (denoted by $h(\xi)$) along a plate edge (line node $\xi \in [-L, L]$ in local coordinates of plate) is given by the following two sets of modified Fourier series

$$h(\xi) = \sum_{\substack{s\in\mathbb{N}\\ l\in\{0,1\}}} H_{ls}\frac{\mathcal{T}_l(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls}L}}\,,\ H_{ls} = \int_{-L}^{L} h(\xi)\frac{\mathcal{T}_l(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls}L}}\mathrm{d}\xi\,, \tag{1a}$$

$$h(\xi) = \sum_{\substack{s\in\mathbb{N}\\ l\in\{0,1\}}} H_{ls}\frac{\mathcal{T}^*_l(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls}L}}\,,\ H_{ls} = \int_{-L}^{L} h(\xi)\frac{\mathcal{T}^*_l(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls}L}}\mathrm{d}\xi\,, \tag{1b}$$

where $\mathbb{N} = \{0, 1, 2, ...\}$ is the non-negative integer set, and the subscript 'l', taking value of either '0' or '1', denotes the corresponding symmetric or antisymmetric functions (and coefficients). Here, ζ_{ls} is given as $\zeta_{ls} = 2$ when $l = 0, s = 0$ and otherwise $\zeta_{ls} = 1$. The corresponding modified Fourier basis function $\mathcal{T}_l(\gamma_{ls}\xi)$ and $\mathcal{T}_l^*(\gamma_{ls}\xi)$ in Eq. (1) is defined as

$$\mathcal{T}_l(\gamma_{ls}\xi) = \begin{cases} \cos(\frac{s\pi}{L}\xi) & l = 0 \\ \sin\left((s+\frac{1}{2})\frac{\pi}{L}\xi\right) & l = 1 \end{cases}, \quad \mathcal{T}_l^*(\gamma_{ls}\xi) = \begin{cases} \sin(\frac{s\pi}{L}\xi) & l = 0 \\ \cos\left((s+\frac{1}{2})\frac{\pi}{L}\xi\right) & l = 1 \end{cases} \tag{2}$$

with $\xi \in [-L, L]$, $s \in \mathbb{N}$. The above two sets of modified Fourier series provide complete and orthogonal sets to described any one-dimensional function $h(\xi)$ of Eq. (1). It should be emphasised that the above modified Fourier series has strong orthogonality which is one of the most important factors that makes the SDSM numerically stable with no precondition, therefore any higher order of modified Fourier series can be adopted in the computation to compute results within any desired accuracy.

By using the above modified Fourier series, the general solution of the governing differential equation (GDE) for out-of-plane [10] and inplane [13] vibration of plate elements with arbitrary boundary conditions in the frequency domain can be achieved. In the next step, the SDS matrix for an element can be analytically formulated by substituting the above general solution into the general boundary conditions (BC) by some algebraic manipulation. Indeed, the analytical expressions involved in the SDSM are concise but can be used to handle complex structures with any arbitrary boundary conditions [10–12].

Next, the analytically expressed spectral dynamic stiffness (SDS) matrix of elements can be assembled directly to model complex structures. The assembly procedure is similar to that of the finite element (FE) method with the exception that the FE elements are generally connected at point nodes whereas the SDS elements are connected on *line nodes*. Here the line nodes represent either the plate boundaries and/or the inter-element edges which have the flexibility to describe any arbitrary BC or continuity conditions (either classical [10–13] or non-classical [14, 15]). In general, for an assembly structure, the analytical SDSM formulation can be written in the following form

$$\boldsymbol{f} = \boldsymbol{K}\boldsymbol{d}\,, \tag{3}$$

where $\boldsymbol{K}$ is the SDS matrix of the complete assembly structure, which relates the modified Fourier coefficient vector of the force $\boldsymbol{f}$ to that of the displacement $\boldsymbol{d}$ on all of the line nodes (boundaries and inter-element edges) of the assembly structure, so that

$$\boldsymbol{f} = \left[\boldsymbol{f}_1^T, \boldsymbol{f}_2^T, \cdots, \boldsymbol{f}_i^T, \cdots, \boldsymbol{f}_{N_{tlDOF}}^T\right]^T, \quad \boldsymbol{d} = \left[\boldsymbol{d}_1^T, \boldsymbol{d}_2^T, \cdots, \boldsymbol{d}_i^T, \cdots, \boldsymbol{d}_{N_{tlDOF}}^T\right]^T. \tag{4}$$

In Eq. (4), the subscript N_{tlDOF} is the total number of *line degrees of freedom* (*line DOF*) of the plate assembly (Theoretically, each *line DOF* has infinite DOF since each BC function is a continuous function on $\xi \in [-L, L]$). Here, $N_{tlDOF} = ln \times N_{lDOF}$ where ln is the number of total *line nodes* of the plate assembly whereas N_{lDOF} represents the number of *line DOF* of each line node (for instance, an individual rectangular Kirchhoff plate, being a special case of the assembly, has four edges, i.e., $ln = 4$ and each edge has two *line DOF* W and ϕ, i.e., $N_{lDOF} = 2$). Each force $\boldsymbol{f}_i$ and displacement $\boldsymbol{d}_i$ sub-vectors in Eq. (4) take the following form

$$\boldsymbol{f}_i = [F_{i00}, F_{i01}, F_{i02}, \cdots, F_{i10}, F_{i11}, F_{i12}, \cdots]^T, \tag{5a}$$

$$\boldsymbol{d}_i = [D_{i00}, D_{i01}, D_{i02}, \cdots, D_{i10}, D_{i11}, D_{i12}, \cdots]^T, \tag{5b}$$

where F_{ils} and D_{ils} ($l \in \{0, 1\}, s \in \mathbb{N}$) are respectively the modified Fourier coefficients of the corresponding force $f_i(\xi)$ and displacement $d_i(\xi)$ BC (or CC) applied on the ith *line DOF* of the assembly, which are obtained by applying Eq. (1) onto $f_i(\xi)$ and $d_i(\xi)$ respectively to give

$$F_{ils} = \int_{-L}^{L} f_i(\xi) \frac{\mathcal{T}_l(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls} L}} \mathrm{d}\xi \,, \quad D_{ils} = \int_{-L}^{L} d_i(\xi) \frac{\mathcal{T}_l(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls} L}} \mathrm{d}\xi \,, \tag{6a}$$

$$\text{or} \quad F_{ils} = \int_{-L}^{L} f_i(\xi) \frac{\mathcal{T}_l^*(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls} L}} \mathrm{d}\xi \,, \quad D_{ils} = \int_{-L}^{L} d_i(\xi) \frac{\mathcal{T}_l^*(\gamma_{ls}\xi)}{\sqrt{\zeta_{ls} L}} \mathrm{d}\xi \,. \tag{6b}$$

Therefore, each term of either F_{ils} or D_{ils} in Eq. (6) represents a frequency-wavenumber dependent DOF (FWDOF) of the ith *line DOF*. In this way, the BC (or CC) can be arbitrarily prescribed along any line DOF, which are directly transformed through Eq. (6) into vector form (i.e., $\boldsymbol{f}_i$ and $\boldsymbol{d}_i$) of Eq. (5) and eventually into $\boldsymbol{f}$ and $\boldsymbol{d}$ in Eq. (4).

In essence, the elegance of the SDSM lies in representing a dynamical system very accurately by using an extremely small number of DOF in an analytical and concise manner. This makes the SDSM superior to other numerical or analytical methods in terms of both accuracy and computational efficiency within low, medium and high frequency ranges. The merits of the SDSM are exploited by the application of the well-known Wittrick-Williams (WW) algorithm [16] which is further enhanced by some techniques as described in this section. Suppose that ω denotes the circular (or angular) frequency of a vibrating structure, then according to the WW algorithm, as ω is increased from zero to ω^*, the number of natural frequencies passed (J) is given by

$$J = J_0 + s\{\boldsymbol{K}_f\} \,, \tag{7}$$

where $s\{\boldsymbol{K}_f\}$ corresponds to the negative inertia of the final SDS matrix $\boldsymbol{K}_f$ evaluated at $\omega = \omega^*$; and J_0 is given by

$$J_0 = \sum_m J_{0m} \,, \tag{8}$$

where J_{0m} is the number of natural frequencies between $\omega = 0$ and $\omega = \omega^*$ for an individual component member when its boundaries are fully clamped. For more details, interested readers are referred to [10, 12, 16]. A similar strategy described in [10] is also adopted here to provide an efficient and reliable prediction for the above J_{0m} which is based on the closed-form solution of each members subject to full simple supports. Therefore, J_{0m} of Eq. (8) can be obtained by applying the WW algorithm in reverse to give

$$J_{0m} = J_{Sm} - s(\boldsymbol{K}_{Sm}) \,, \tag{9}$$

where J_{Sm} is the overall mode count of a certain member with all boundaries subject to simple supports, and $s(\boldsymbol{K}_{Sm})$ is the sign count of its formulated SDS matrix $\boldsymbol{K}_{Sm}$. First, the computation of J_{Sm} in Eq. (9) is accomplished in an analytical manner by solving a number theory problem. Next, the computation of $s(\boldsymbol{K}_{Sm})$ in Eq. (9) is achieved in an elegant way by taking advantage of the mixed-variable formulation, e.g., see **Ref**. The above two techniques of computing J_{Sm} and $s(\boldsymbol{K}_{Sm})$ resolves with conclusive certainty the problem of determining J_0 in a highly efficient, accurate and reliable manner.

The SDSM [10–15] has some important properties which can be summarised as follows:

- **Exactness:** The SDSM should be regarded as an exact series-based method which converges to exact results with an exceptionally fast convergence rate [10, 11]. This is because the formulation satisfies exactly the GDE of structure vibration and any arbitrary BC are satisfied in an exact series sense. Moreover, unlike most other analytical methods, the SDSM is unconditionally numerically stable for any higher order series terms, allowing the method to compute results within any desired accuracy.
- **Efficiency:** The SDSM is highly efficient mainly due to the fact that it uses a very small number of DOF which, nevertheless, represent the structure most accurately. This is because the spectral dynamic stiffness (SDS) matrix is formulated on the line nodes (similar to the boundary element

method), and represents the system in a spectral sense. Moreover, the enhancements of the Wittrick-Williams algorithm where the so-called J_m problem has been elegantly resolved to allow modelling of complex structures with as few elements as possible.

- **Robustness:** The SDSM computes any required natural frequencies of complex structures covering from low to high frequency ranges. There is no possibility of missing any natural frequencies and no spurious frequencies will be unnecessarily captured. This is also due to the application of the enhanced Wittrick-Williams algorithm.
- **Versatility:** The SDSM can be assembled as easily as the finite element method. Clearly, the SDSM can handle not only a single plate but also complex structures made of isotropic as well as composite materials which can be subjected to any arbitrary boundary conditions [10–12, 15], which also includes arbitrary non-uniform elastic supports, mass attachments and coupling constraints [13, 14].

3. Results

Table 1: The results are presented for the frequency parameter $\lambda = 4\omega a^2/\pi^2\sqrt{\rho h/D}$ using SDSM for four cases from low (1-10th modes), and medium (20-100th modes) to high (200-1000th modes) frequencies. The first three cases are for isotropic square plates with different BCs whereas the final case is for a CSCC square isotropic plate with internal line support located at three-tenth ($x/(2a) = 0.3$) of the edge.

Mode	FFFF	CCCC		CSCS			CSCC with internal line support		
Methods	**SDSM**	**SDSM**	DSC[a]	**SDSM**	Levy[b]	DSC[a]	**SDSM**	DSC	Ritz(trig.)
M, N	**25+25**	**25+25**	101×101	**30+30**	Exact	101×101	**2(25+25)**	101×101	50×50
1	**0.00000**	**3.64606**	3.6461	**2.93333**	2.9333	2.9333	**4.90797**	4.9080	4.9080
2	**0.00000**	**7.43635**	7.4364	**5.54663**	5.5466	5.5466	**7.89990**	7.8999	7.9000
3	**0.00000**	**7.43635**	7.4364	**7.02430**	7.0243	7.0243	**12.1929**	12.1929	12.1932
4	**1.36461**	**10.9646**	10.9647	**9.58349**	9.5835	9.5835	**13.1294**	13.1295	13.1297
5	**1.98550**	**13.3319**	13.3320	**10.3567**	10.3567	10.3567	**15.1607**	15.1608	15.1612
6	**2.45909**	**13.3951**	13.3952	**13.0801**	13.0801	13.0802	**20.2351**	20.2353	20.2360
7	**3.52607**	**16.7180**	16.7182	**14.2057**	14.2057	14.2057	**20.4862**	20.4863	20.4870
8	**3.52607**	**16.7180**	16.7182	**15.6821**	15.6821	15.6821	**20.5496**	20.5498	20.5509
9	**6.19004**	**21.3303**	21.3305	**17.2597**	17.2597	17.2597	**23.2337**	23.2340	23.2356
10	**6.19004**	**21.3303**	21.3305	**20.2450**	20.2450	20.2451	**25.8856**	25.8860	25.8879
20	**15.4864**	**37.6251**	37.6257	**34.9090**	34.9090	34.9093	**41.3478**	41.3481	41.3507
40	**36.8797**	**68.6660**	68.6682	**64.5845**	64.5845	64.5857	**72.8807**	72.8826	72.8925
60	**56.8884**	**94.3824**	94.3854	**91.1837**	91.1837	91.1863	**105.022**	105.027	105.050
80	**81.1448**	**126.705**	126.712	**118.971**	118.971	118.975	**134.086**	134.092	134.120
100	**106.007**	**156.649**	156.660	**147.991**	147.991	147.993	**160.517**	160.530	160.586
200	**219.427**	**290.220**	290.254	**281.655**	281.655	281.668	**306.030**	306.046	306.182
400	**462.626**	**561.364**	561.465	**550.555**	550.555	550.576	**580.000**	580.096	580.640
600	**704.645**	**826.952**	827.253	**814.177**	814.177	814.316	**846.353**	846.713	848.265
800	**951.601**	**1092.55**	1092.95	**1074.18**	1074.18	1074.41	**1113.88**	1114.19	1116.31
1000	**1197.00**	**1349.29**	1349.98	**1333.87**	1333.87	1334.31	**1380.36**	1381.06	1384.32
Time (s)	**0.37**	**0.35**	–	**6.07**	–	–	**12.0**	–	–

[a] Discrete Singular Convolution method [17]
[b] Levy solution (exact) [17]
[c] Discrete Singular Convolution method [18]
[d] Trigonometric Ritz method [19]

The SDSM provides exact solutions with an extremely fast computation speed. This is a tremendous advantage which applies to individual composite plates (e.g., FFFF, CCCC, CSCS cases in Table 1) as well as composite plate assemblies (CSCC with internal support). (The notation comprising four letters successively represent the right, up, left and bottom edges respectively in an anticlockwise sense.) Table 1 shows the results for four different cases with the first three cases modelled by one SDS element whereas the fourth one modelled by two SDS elements. All SDSM results are presented with accuracy up to six significant figures. The number of terms used in the series expansion (M and N) are indicated in the table. Also, the total computation time taken by the SDSM for each case is recorded in the last column of Table 1. The results computed by the current method are compared with those obtained by other methods including the discrete singular convolution (DSC) method [17, 18], exact Levy solution [17] and Trigonometric Ritz method [19]. It is evident from Table 1 that the SDSM behaves extraordinarily well in terms of accuracy and computational efficiency not only for the low frequency range, but also for the medium to high frequency ranges. With no more than 30 terms taken in the series solution, all of the 20 natural frequencies covering low to high frequency ranges have six significant digit accuracy for all of the four cases. It can be seen that all SDSM results for the CSCS case coincide with exact Levy solutions [17] as expected. Furthermore, those 20 highly accurate results for each case were obtained within 12 seconds in total. In particular, each of the FFFF and CCCC cases took less than 0.4 second which is indeed, a remarkable computational efficiency.

Similarly, Table 2 shows the inplane free vibration analysis results computed by the SDSM with remarkable accuracy and computational efficiency. The SDSM is used to predict the medium (10th-100th) and high (200th-1000th) natural frequencies for four cases. When using the SDSM, only one SDS element is used in the modelling with both M and N adopted as 20, and all SDSM results are given with accuracy of six significant figures. The results computed by SDSM are compared with FE solutions computed by ANSYS (FEM2, using a 300×300 mesh of Plane 182 elements) with only three significant figures for the 10th-100th modes (results with two significant figures are computed when a coarser mesh FEM1 100×100 is used). Higher natural modes are not computed by the FEM since the solvers provided in the FEM becomes highly inefficient and unreliable for higher modes. The final

Table 2: Dimensionless natural frequency parameters $\lambda = 2\omega a/\pi\sqrt{\rho/G}$ for inplane free vibration plates. The SDSM is applied to compute 11 natural frequencies of the four cases covering medium (10th-100th) to higher (200th-1000th) modes. Side by side are the finite element solutions obtained by ANSYS using a very fine mesh (FEM2, 300×300) of Plane 182 elements (a coarser mesh 100×100 FEM1 is also adopted for the $S_2S_1S_2S_1$ case only). Only medium (10th-100th) modes are given for the FE solutions. The final matrix size and the total computational time of both the SDSM and FEM are given in the last two rows.

Mode	CCCC		$S_2S_1S_2S_1$			$CCCS_2$		CCS_2S_2	
	SDSM	FEM2	SDSM	FEM1	FEM2	SDSM	FEM2	SDSM	FEM2
10	**3.82718**	3.82859	**3**	3.00111	3.00009	**3.72833**	3.72845	**3.56133**	3.56150
20	**5.17103**	5.17542	**4.12311**	4.12595	4.12342	**5.07830**	5.07865	**4.93554**	4.93602
40	**6.91510**	6.92134	**6.08276**	6.09196	6.08379	**6.80026**	6.80100	**6.59543**	6.59643
60	**8.38312**	8.38503	**7.28011**	7.29566	7.28172	**8.22259**	8.22479	**8.01245**	8.01452
80	**9.36414**	9.36610	**8.60233**	8.62697	8.60544	**9.31753**	9.31987	**9.22457**	9.22677
100	**10.5505**	10.5549	**9.84886**	9.86266	9.85316	**10.3622**	10.3655	**10.1912**	10.1933
200	**14.5867**	–	**13.6277**	–	–	**14.4709**	–	**14.2358**	–
400	**20.2866**	–	**19.2725**	–	–	**20.1682**	–	**20.0177**	–
600	**24.5966**	–	**23.7697**	–	–	**24.4609**	–	**24.3223**	–
800	**28.2882**	–	**27.4591**	–	–	**28.1562**	–	**27.9424**	–
1000	**31.5344**	–	**30.6128**	–	–	**31.3854**	–	**31.2660**	–
Sign. Fig.	**6**	3	**6**	2	3	**6**	3	**6**	3
Matrix Size	**39**	1.80E+05	**158**	2.00E+04	1.80E+05	**39**	1.80E+05	**78**	1.80E+05
Time (s)	**0.25**	24.6	**1.25**	5.00	23.5	**1.06**	27.5	**1.11**	23.0

matrix size and total computational time for both methods are also given in the last two rows of Table 2. To solve the tabulated 11 modes with six significant figures covering medium to high frequency ranges, the SDSM took only 0.25-1.25 s; whereas the well-developed FEM package `ANSYS` took 23-28 s but compute only the first six medium modes with three significant figures. It is apparent that SDSM is far more superior to the FEM in free vibration analysis within medium to high frequency ranges. The major advantage of the current SDSM lies in the fact that the SDS formulation satisfies the GDE exactly and uses extremely low number of DOF to represent the system most accurately. For the four case studies shown in Table 2, the final matrix size of the SDSM was only 39-158, which is in a sharp contrast to the FEM using as many as 1.80E05 DOF. This great advantage establishes the SDSM as an ideal tool for parametric and optimisation studies, not only within low frequency range but also within medium to high frequency ranges.

Apart from the excellent computational efficiency and accuracy, the SDSM is superior to other analytical (or semi-analytical) methods due to its versatility. Associated theories have been developed to model arbitrarily non-uniform elastic supports, mass attachments and elastic coupling constraints, which has further broadened the application scope of the SDSM. In [14], the modal analysis of a Goland wing model has been performed by using the SDSM. In this model (see Fig. 1), the presence of stores, engines as well as flap affects the dynamic properties of the aircraft significantly and therefore, influences the aeroelastic stability as well as the maneuverability of the aircraft in operation. The present research proposes a novel analytical method for modal analysis of composite rectangular wing with uniform or non-uniform attachments like engine as well as elastically coupled flap in an efficient and elegant way.

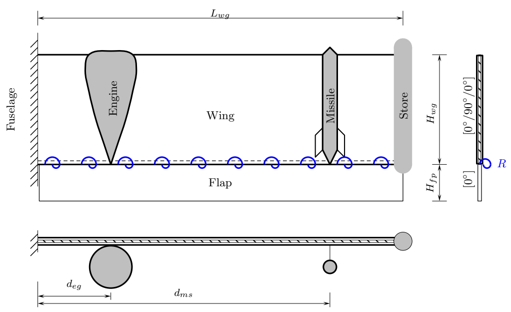

Figure 1: A composite Goland wing with rotationally coupled flap as well as engine, missile and store attachments.

4. Conclusions

A recently developed Spectral Dynamic Stiffness Method (SDSM) has been presented in this paper. The methodology and advantages are detailed which is followed by numerical results and applications. It is demonstrated that the SDSM provides exact solutions with remarkable efficiency. Moreover, the method is versatile for structures with arbitrary boundary conditions and with complex geometries, which has open novel possibilities in structural dynamic analysis.

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