INFLUENCE OF BOUNDARY RESTRAINING STIFFNESS ON VIBRATION CHARACTERISTICS OF CONICAL SHELL STRUCTURE MADE OF FUNCTIONALLY GRADED MATERIAL

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As one of the most important structures, conical shells are widely used in various occasions, such as aerospace engines and ocean engineering. In this paper, a simulation model of conical shell made of the functionally gradient material is established to study the Influence of boundary restraint on its characteristics. Spring stiffness can be adjusted to simulate various boundary conditions. The vibrating displacement fields are constructed as the Fourier series with supplementary functions with the consideration of boundary equilibrium requirement. All the unknown can be solved from a standard eigenvalue problem, which is obtained from energy formulation in conjunction with Rayleigh-Ritz method. Several numerical results are then presented to show the correctness and efficiency of the current model. The influence of boundary restraining stiffness, parameters of functionally gradient materials on the modal characteristic are then discussed in detail. Finally, some concluding remarks are made.

Keywords: FGM Conical Shells, natural frequency, Boundary Conditions

1. Introduction

The conical shells have been widely used in various engineering fields and intensively in aerospace structures, for example, the noses of supersonic aircraft and missiles. FGMs now have been considered as one of the most promising candidates for future smart composites in numerous engineering fields. FGM conical shells are made from a mixture of ceramics and metals and are more characterized by a continuous and smooth change of the mechanical properties from one surface to another. In the service life, the FGM conical shells are often subjected to dynamic loadings. Therefore, the free vibration characteristic of the FGM conical shells must be studied for safety and stability reasons.

The development of conical shell structures have been considerable and well established. Niordson [1] used energy method to solve conical shells bucking problem. Mustari and Sachenkov [2] derived the single Galerkin method to analyse the vibration and the stability of structure shells under various boundary conditions. Singer [3]-[4] studied the bucking problem of conical shells and cylinder shells under different boundary conditions. Daninel [5] gave the vibration characteristic of conical shells with a clamped end a free end by Rayleigh-Ritz method. Li and Lam [6]-[7] has done a lot of research about the static and dynamic characteristics of conical shells and did some analyses of free vibration characteristics of conical shell and the influence of boundary conditions and various parameters on the free vibration frequency.

In this paper, an FGM conical shell was treated as the research object and the characteristic equation of the system was derived by using Rayleigh-Ritz method based on the expressions of kinetic and potential energy. The vibrating displacement fields are constructed as the Fourier series with supplementary functions with the consideration of boundary equilibrium requirement. Then characteristic equations in different boundary conditions were solved by using MATLAB programming language. Then eigenvalues were gotten respectively and some comparisons were done with references. Two frequencies matched well, so the method used in this paper verified right. At last, the influence of boundary restraining stiffness, parameters of functionally gradient materials on the modal characteristic are then discussed in detail. Finally, some concluding remarks are made.

2. Theoretical formulations

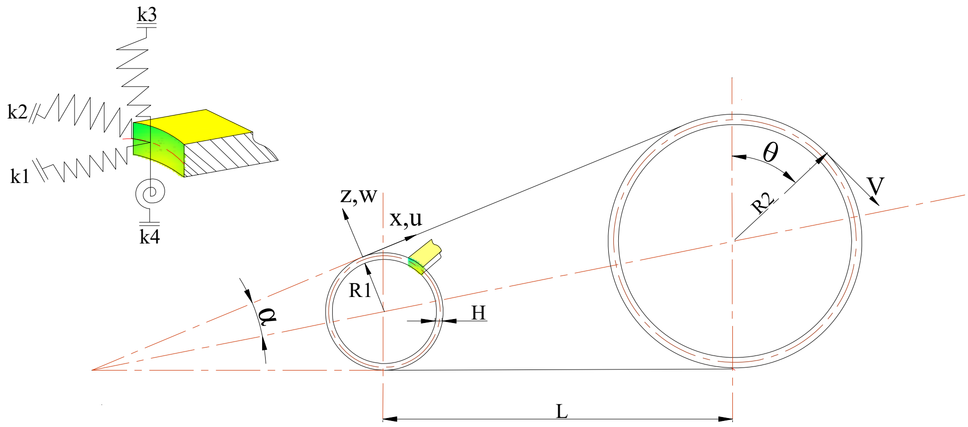

Fig.1: Modeling of the shell and its boundary conditions

Figure 1 shows the elastic restraints along the left end of a FGM conical shell of small end radius R_1, big end radius R_2, thickness H and length L. The variables k_1, k_2, k_3 denote the stiffnesses for the linear springs, and k_4 is the stiffness for the rotational spring. Similarly, a set of springs (k_5, k_6, k_7 and k_8) can also be applied to the right end. All the classical homogeneous boundary conditions can be considered as the special cases when the stiffness for each of these springs is either extremely large or sufficiently small.

Functionally graded materials are composed of two constituent materials, the variations through the thickness of Young's modulus E, Poisson ratio μ and the mass density per unit volume ρ can be written as [8]:

$$\begin{cases} E=\left(E_o-E_i\right)\left(\dfrac{2z+H}{2H}\right)^N+E_i \\ \mu=\left(\mu_o-\mu_i\right)\left(\dfrac{2z+H}{2H}\right)^N+\mu_i \\ \rho=\left(\rho_o-\rho_i\right)\left(\dfrac{2z+H}{2H}\right)^N+\rho_i \end{cases} \tag{1}$$

where z is the thickness coordinate ($-H/2 \le z \le H/2$), and N is the gradient index. Subscripts i and o refer to the metal and ceramic constituents, respectively. When the value $N=0$, a fully ceramic shell is intended and infinite$N$, a fully metallic shell. When the gradient index is increased, the content of metal in the FGM layers decreased.

The vibrating displacement fields constructed as the Fourier series with supplementary functions are shown as:

$$u=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\cos\lambda_m\cos n\theta+\sum_{n=0}^{\infty}\left(a_n\xi_1(x)+b_n\xi_2(x)\right)\cos n\theta \tag{2}$$

$$v=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{mn}\cos\lambda_m\sin n\theta+\sum_{n=0}^{\infty}\left(c_n\xi_1(x)+d_n\xi_2(x)\right)\sin n\theta \tag{3}$$

$$w=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}\cos\lambda_m\cos n\theta+\sum_{n=0}^{\infty}\left(e_n\zeta_1(x)+f_n\zeta_2(x)+g_n\zeta_3(x)+h_n\zeta_4(x)\right)\cos n\theta \tag{4}$$

Based on the Donnell-Mushtari theory, the surface strains and curvatures are defined as:

$$\begin{aligned}&e_1=\frac{\partial u}{\partial x},e_2=\frac{1}{R(x)}\frac{\partial v}{\partial\theta}+\frac{u\,sin\,\alpha}{R(x)}+\frac{w\,cos\,\alpha}{R(x)},\gamma=\frac{\partial v}{\partial x}+\frac{1}{R(x)}\frac{\partial u}{\partial\theta}-\frac{v\,sin\,\alpha}{R(x)}\\&\kappa_1=-\frac{\partial^2 w}{\partial x^2},\kappa_2=-\frac{1}{R^2(x)}\frac{\partial^2 w}{\partial\theta^2}-\frac{sin\,\alpha}{R(x)}\frac{\partial w}{\partial x},\tau=-\frac{1}{R(x)}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{sin\,\alpha}{R^2(x)}\frac{\partial w}{\partial\theta}\end{aligned} \tag{5}$$

The energy expressions of conical shell include the kinetic energy expressions and the potential energy expressions:

$$T=\frac{1}{2}\rho_t\int_0^L\int_0^{2\pi}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]R(x)\,\mathrm{d}\theta\mathrm{d}x \tag{6}$$

$$\begin{aligned}V=&\frac{1}{2}\int_0^{2\pi}\int_0^L\left[\{\varepsilon\}^T[S]\{\varepsilon\}\right]R(x)\,dxd\theta\\&+\frac{1}{2}\int_0^{2\pi}\left\{\left(k_1u^2+k_2v^2+k_3w^2+K_4\left(\frac{\partial w}{\partial x}\right)^2\right)_{x=0}+\left(k_5u^2+k_6v^2+k_7w^2+K_8\left(\frac{\partial w}{\partial x}\right)^2\right)_{x=L}\right\}Rd\theta\\=&\frac{1}{2}\int_0^{2\pi}\int_0^L\begin{bmatrix}A_{11}e_1^{\,2}+A_{22}e_2^{\,2}+2A_{12}e_1e_2+D_{11}\kappa_1^{\,2}+D_{22}\kappa_2^{\,2}\\+2D_{12}\kappa_1\kappa_2+2B_{11}e_1\kappa_1+2B_{12}e_1\kappa_2+2B_{12}e_2\kappa_1\\+2B_{22}e_2\kappa_2+A_{66}\gamma^2+4D_{66}\tau^2+4B_{66}\gamma\tau+A_{66}e_{23}^{\,2}\end{bmatrix}Rdxd\theta\\&+\frac{1}{2}\int_0^{2\pi}\left\{\left(k_1u^2+k_2v^2+k_3w^2+K_4\left(\frac{\partial w}{\partial x}\right)^2\right)_{x=0}+\left(k_5u^2+k_6v^2+k_7w^2+K_8\left(\frac{\partial w}{\partial x}\right)^2\right)_{x=L}\right\}Rd\theta\end{aligned} \tag{7}$$

Then do derivation, transformation, substitution to the energy equations by the Rayleigh-Ritz method based on Hamilton equation:

$$\delta\int_{t_0}^{t_1}(V-T)\,dt=0 \tag{8}$$

By substituting Eqs. (1) - (7) into Eq. (8), the final system equations about the Fourier coefficients can be obtained as

$$\begin{bmatrix}\mathrm{K}^{uu}&\mathrm{K}^{uv}&\mathrm{K}^{uw}\\\mathrm{K}^{uv^T}&\mathrm{K}^{vv}&\mathrm{K}^{vw}\\\mathrm{K}^{uw^T}&\mathrm{K}^{vw^T}&\mathrm{K}^{ww}\end{bmatrix}\begin{Bmatrix}\overline{\mathrm{U}}\\\overline{\mathrm{V}}\\\overline{\mathrm{W}}\end{Bmatrix}-\omega^2\begin{bmatrix}\mathrm{M}^{uu}&0&0\\0&\mathrm{M}^{vv}&0\\0&0&\mathrm{M}^{ww}\end{bmatrix}\begin{Bmatrix}\overline{\mathrm{U}}\\\overline{\mathrm{V}}\\\overline{\mathrm{W}}\end{Bmatrix}=0 \tag{9}$$

where

$$\rho_t=\int_{-H/2}^{H/2}\rho\,\mathrm{d}z \tag{10}$$

$$\overline{\mathrm{U}}=\left\{A_{00},A_{01},\cdots,A_{mn},\cdots,A_{MN},a_0,a_1,\cdots,a_n,\cdots,a_N,b_0,b_1,\cdots,b_n,\cdots,b_N\right\}^T \tag{11}$$

$$\overline{\mathrm{V}}=\left\{B_{00},B_{01},\cdots,B_{mn},\cdots,B_{MN},c_0,c_1,\cdots,c_n,\cdots,c_N,d_0,d_1,\cdots,d_n,\cdots,d_N\right\}^T \tag{12}$$

$$\begin{aligned}\overline{\mathrm{W}}=\{&C_{00},C_{01},\cdots,C_{mn},\cdots,C_{MN},e_0,e_1,\cdots,e_n,\cdots,e_N,f_0,f_1,\cdots,\\&f_n,\cdots,f_N,g_0,g_1,\cdots,g_n,\cdots,g_N,h_0,h_1,\cdots,h_n,\cdots,h_N\}^T\end{aligned} \tag{13}$$

3. Numerical results and discussion

According to the system characteristic equation, the eigenvalue (ω) for different boundary conditions can be solved by MATLAB programs. The variation through the thickness of Young's modulus

E and mass density per unit volume ρ are the same as Eq. (1). The structural parameters are H=0.004m, R_2=0.4m, $L\sin\alpha/R_2$=0.25, E_o =380Gpa, ρ_o=3800kg/m^3, μ_o=0.3, and E_i=70Gpa, ρ_i=2710kg/m^3, μ_i=0.3。

3.1 Classical boundary conditions

Here some natural frequencies with different gradient index will be presented, and be compared with frequencies from reference [9].

Table 1 Comparisons of natural frequency f for the conical shell with S-S boundaries ($m=1, N=0$)

α	n	present	reference	Error (%)
30^o	1	3882.17	3874	0.2105
	2	3499.85	3516	0.4614
	3	3056.46	3093	1.1954
	4	2667.45	2677	0.3580
	5	2342.08	2331	0.4753
	6	2073.56	2088	0.6916
45^o	1	3393.39	3316	2.0549
	2	3223.41	3187	0.4322
	3	3006.47	3007	0.8988
	4	2797.40	2814	0.9072
	5	2640.38	2642	0.0613
	6	2529.75	2523	0.2675
60^o	1	2688.63	2679	0.3595
	2	2665.32	2645	0.7682
	3	2630.23	2601	1.1237
	4	2522.85	2563	1.5665
	5	2530.96	2549	0.7077
	6	2557.21	2574	0.6523

In table 1, the natural frequencies getting from the present method have been compared with the ones from the reference [9] in the same boundary conditions, and the error is showed clearly. From the errors, it is easy to find that the maximum error is 2.0549%. Most of the errors are about 0.1%--2%, so the present method can be verified to be right in classical boundary conditions. The table 1 also shows the variations of the natural frequencies (Hz) with the circumferential wave numbers n and semi-vertex cone angle for FGM conical shell.

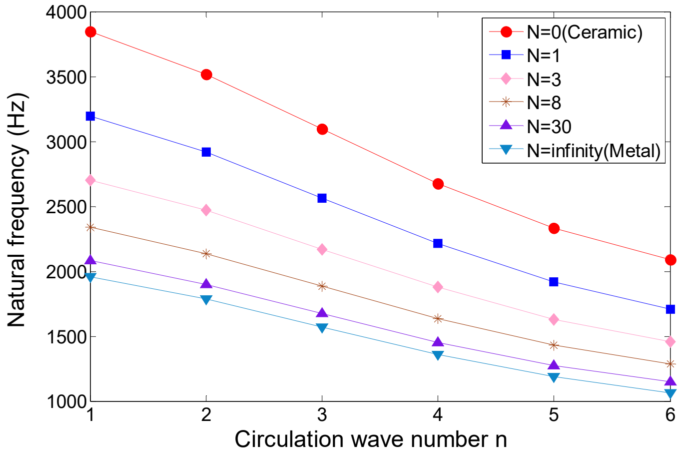

Fig. 1: Natural frequencies associated with various gradient index for the case of semi-vertex cone angle $\alpha=30^o$, m = 1

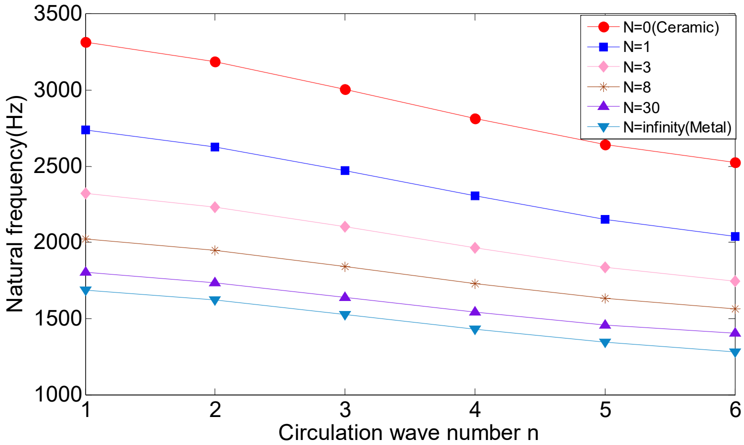

Fig. 2: Natural frequencies associated with various gradient index for the case of semi-vertex cone angle α = 45º, m = 1

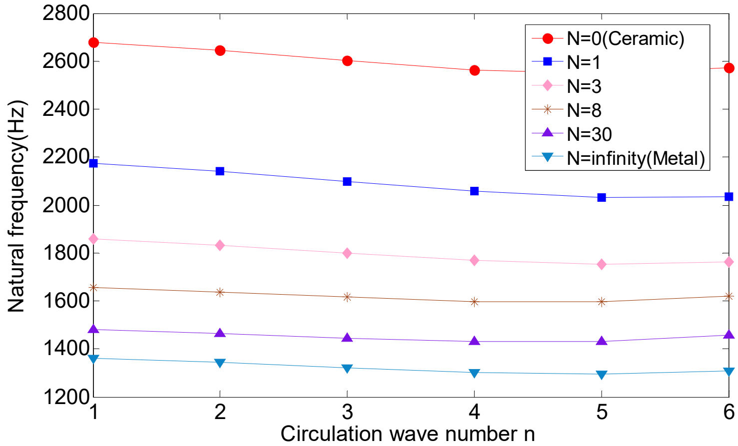

Fig. 3: Natural frequencies associated with various gradient index for the case of semi-vertex cone angle α= 60º, m = 1

Fig. 1- Fig. 3 show the variations of natural frequencies (Hz) with the circumferential wave numbers n for FGM conical shell. The curve N=0, N=infinity show the natural frequencies for a ceramic conical shell and a metal conical shell, respectively. The effects of changing gradient index (N) can be seen from figures. As N increased, the natural frequencies decreased. When N is small, the natural frequencies approached those of Ceramic and when N is large they approached those of Metal. Hence, the natural frequencies for N>0 fell between those of Ceramic and Metal for a given circumferential wave number n.

3.2 Other boundary conditions

In order to find out the relationships between the spring stiffnesses and the frequency, at both two ends of the shell, the elastic supports are set to be only one spring, and the non-dimensional spring stiffness changes from 10^{-4} to 10^{4}. In each case, five frequencies are recorded and then show them in tables as follows:

Table 2 Impact on the natural frequencies of the spring stiffness of $k_1^{'}$

N	$k_1^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	3280.256	2506.727	1895.963	1478.247	1224.229
	10^{-2}	3280.352	2506.869	1896.156	1478.467	1224.440
	10^{0}	3288.300	2518.835	1912.557	1497.353	1242.730
	10^{2}	3326.236	2579.497	2000.768	1606.080	1357.515
	10^{4}	3328.507	2583.328	2006.639	1613.773	1366.303

2	10^{-4}	2743.946	2097.306	1586.201	1235.940	1021.386
	10^{-2}	2744.031	2097.425	1586.360	1236.119	1021.555
	10^{0}	2751.112	2107.430	1599.844	1251.495	1036.321
	10^{2}	2783.708	2157.322	1672.095	1340.468	1130.044
	10^{4}	2785.598	2160.427	1676.889	1346.796	1137.302

Table 3 Impact on the natural frequencies of the spring stiffness of $k_2^{'}$

N	$k_2^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	3280.268	2506.751	1895.991	1478.277	1224.259
	10^{-2}	3281.497	2509.232	1898.966	1481.442	1227.365
	10^{0}	3368.343	2697.700	2128.285	1724.577	1465.205
	10^{2}	3546.680	3207.822	2804.373	2444.435	2167.509
	10^{4}	3551.339	3223.216	2826.208	2467.653	2189.986
2	10^{-4}	2743.955	2097.326	1586.225	1235.966	1021.412
	10^{-2}	2744.984	2099.429	1588.774	1238.718	1024.164
	10^{0}	2817.477	2258.775	1784.580	1448.930	1233.068
	10^{2}	2964.239	2683.360	2349.183	2051.510	1822.564
	10^{4}	2968.017	2695.877	2366.869	2070.166	1840.434

Table 4 Impact on the natural frequencies of the spring stiffness of $k_3^{'}$

N	$k_3^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	3280.348	2507.039	1896.502	1497.059	1225.314
	10^{-2}	3287.807	2534.442	1944.586	1551.196	1320.521
	10^{0}	3317.244	2708.989	2290.788	2053.176	1922.278
	10^{2}	3318.685	2721.242	2316.978	2088.437	1959.168
	10^{4}	3318.699	2721.371	2317.253	2088.805	1959.549
2	10^{-4}	2744.021	2097.567	1586.651	1236.616	1022.288
	10^{-2}	2750.108	2120.533	1626.989	1297.058	1101.979
	10^{0}	2774.262	2271.662	1931.383	1739.625	1631.391
	10^{2}	2775.451	2282.537	1955.186	1771.670	1664.577
	10^{4}	2775.463	2282.651	1955.437	1772.004	1664.918

Table 5 Impact on the natural frequencies of the spring stiffness of $k_4^{'}$

N	$k_4^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	3282.631	2506.733	1896.562	1480.841	1229.890
	10^{-2}	3283.449	2506.736	1896.950	1482.990	1235.597
	10^{0}	3283.460	2506.736	1896.957	1483.030	1235.713
	10^{2}	3283.461	2506.736	1896.957	1483.030	1235.714
	10^{4}	3283.461	2506.736	1896.957	1483.030	1235.714
2	10^{-4}	2746.081	2097.321	1586.626	1237.908	1025.679
	10^{-2}	2746.847	2097.330	1586.988	1240.152	1031.769
	10^{0}	2746.858	2097.330	1586.944	1240.201	1031.918
	10^{2}	2746.858	2097.330	1586.944	1240.201	1031.920

	10^4	2746.858	2097.330	1586.944	1240.201	1031.920

Table 6 Impact on the natural frequencies of the spring stiffness of $k_5^{'}$

N	$k_5^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	2528.631	1802.760	1325.103	1024.706	848.928
	10^{-2}	2529.064	1803.096	1325.399	1024.979	849.167
	10^{0}	2563.995	1830.475	1349.819	1047.679	869.302
	10^{2}	2702.288	1946.184	1458.116	1154.584	972.111
	10^{4}	2709.273	1952.420	1464.208	1160.926	978.664
2	10^{-4}	2117.170	1510.222	1111.153	859.226	710.302
	10^{-2}	2117.514	1510.489	1111.386	859.439	710.487
	10^{0}	2145.039	1532.108	1130.543	877.164	726.058
	10^{2}	2249.864	1621.648	1214.489	960.117	805.373
	10^{4}	2254.980	162.	1219.163	965.012	810.418

Table 7 Impact on the natural frequencies of the spring stiffness of $k_6^{'}$

N	$k_6^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	2528.670	1802.817	1325.160	1024.761	848.979
	10^{-2}	2533.005	1808.726	1331.098	1030.453	854.249
	10^{0}	2780.807	2199.848	1740.362	1424.594	1219.522
	10^{2}	3086.571	2982.473	2744.235	2452.790	2198.990
	10^{4}	3092.956	3006.318	2778.788	2487.712	2231.927
2	10^{-4}	2117.204	1510.270	1111.202	859.273	710.347
	10^{-2}	2120.847	1515.264	1116.263	864.182	714.956
	10^{0}	2329.234	1845.576	1464.442	1202.763	1032.154
	10^{2}	2586.860	2505.905	2316.090	2081.751	1871.918
	10^{4}	2592.248	2526.009	2345.439	2111.661	1899.849

Table 8 Impact on the natural frequencies of the spring stiffness of $k_7^{'}$

N	$k_7^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	2529.319	1803.601	1326.292	1026.341	851.010
	10^{-2}	2585.933	1875.652	1428.140	1163.832	1021.603
	10^{0}	2791.855	2240.190	1969.752	1857.096	1814.123
	10^{2}	2799.770	2260.212	1999.955	1893.844	1852.496
	10^{4}	2799.851	2260.413	2000.267	1894.221	1852.887
2	10^{-4}	2117.750	1510.933	1112.155	860.590	712.022
	10^{-2}	2165.534	1572.205	1198.749	976.625	854.691
	10^{0}	2339.949	1895.120	1691.910	1614.000	1579.423
	10^{2}	2346.660	1912.811	1720.872	1650.007	1616.718
	10^{4}	2346.728	1912.994	1721.171	1650.378	1617.098

Table 9 Impact on the natural frequencies of the spring stiffness of $k_8^{'}$

N	$k_8^{'}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	10^{-4}	2533.828	1807.590	1333.306	1038.728	869.852

	10^{-2}	2535.327	1809.336	1336.791	1045.670	881.893
	10^{0}	2535.346	1809.360	1336.841	1045.774	882.084
	10^{2}	2535.346	1809.360	1336.841	1045.775	882.086
	10^{4}	2535.346	1809.360	1336.841	1045.775	882.086
	10^{-4}	2121.823	1514.968	1119.235	872.050	727.648
	10^{-2}	2123.164	1516.807	1123.237	880.060	741.087
2	10^{0}	2123.181	1516.833	1123.297	880.191	741.328
	10^{2}	2123.182	1516.833	1123.298	880.192	741.330
	10^{4}	2123.182	1516.833	1123.298	880.192	741.330

From Table 2 –Table 9,it could be found that no matter which spring is set at both two ends, the natural frequency increases with the increasing of the spring stiffness firstly, then reach to a stable value. So the effect of spring stiffness on natural frequency is smaller and smaller when the spring stiffness continue increasing. And with the increasing of the spring stiffness, the natural frequency of the gradient index 0.5 increases more quickly than that of the gradient index 2.

4. Conclusion

A study on the vibration of FGM conical shells has been presented. The analysis was done by Rayleigh-Ritz method and the MATLAB programming language was used to solve the frequency equations in different boundary conditions. For validation, the results are compared with those in the literature and have found to be accurate. So the method used in this paper was proved to be right. And the influence of the gradient index (N) on the frequencies for FGM conical shells has been found, the natural frequencies increases when N increases. Next the natural frequencies were shown in tables, from which the effects of different elastic supports on frequencies were analyzed. The natural frequency increases with the increasing of the spring stiffness firstly, then reach to a stable value.

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