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THE APPLICATION OF RADIAL POINT INTERPOLATION MESHLESS METHOD BASED ON THE HERMITE COLLOCATION METHOD IN THE ACOUSTIC FIELD

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Based on the Hermite collocation method, radial point interpolation meshless method (RPIM) is employed in this paper to study the acoustic problems in frequency domain. From the perspective of stability and error analysis, the reliability and availability of RPIM is investigated to deal with Neumann boundary conditions by adopting multiquadrics Function, Gaussian function and thin plate spline. The scheme of distributing irregular points in the computational domain is proposed to verify the adaptive ability of RPIM based on the Hermite collocation method in solving acoustic problems. The current method is validated through handling with several acoustic problems.

Keywords: Meshless, Radial point interpolation method (RPIM), Hermite collocation, RBF

1. Introduction

In recent decades, meshless methods are involved in various problems in physics as well as engineering field. Meshless methods can be traced back to the smoothed particle hydrodynamics method by Lucy and Gingold in 1977[1, 2]. The basic idea of meshless method is distributing nodes in computational domain and adopting shape function of nodes to approximately represent physical quantity in computational field. Meshless methods not only possess superiority in reliability over traditional mesh method, such as FEM and FVM, but also require simpler pre-treatment, high accuracy and decent computational efficiency. Radial point interpolation method (RPIM) was proposed by Liu [3] and applied to study free vibration problems of 2-D solids. Furthermore, the Hermite-type interpolation is adopted in RPICM to improve the accuracy for solving nonlinear Poisson equation with Neumann boundary conditions [4]. Recently, ArmanShojaei[5] proposed infinite clouds for the analysis of acoustic problems defined on unbounded domains. In this paper, the globally supported RBF collocation method is introduced to solve 1D and 2D Helmholtz equation with Neumann boundary condition and some examples is proposed to verify its higher accuracy and easier implementation.

2. Hermite radial point interpolation method

The approximation of a function $u(x)$ can be written as a linear combination of radial basis functions around x and its normal derivatives on boundaries at the n nodes within its support domain, then n_b nodes on boundaries.

$$u(\mathbf{x}) \cong \tilde{u}(\mathbf{x}) = \sum_{i=1}^{n} a_i R_i(r) + \sum_{j=1}^{n_b} b_j \frac{\partial R_j^b(r)}{\partial n} + \sum_{k=1}^{m} c_k P_k(\mathbf{x}) \tag{1}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, and (x_0, y_0, z_0) is the central point. And $R_i(r)$ are the strictly positive definite radial basis functions (RBF) with coefficients a_i , b_j are the coefficients of the normal derivative of radial basis $R_i(r)$ at the points on boundaries. c_k are the coefficients of the additional polynomial $P_k(\mathbf{x})$ .The coefficients a_i , b_j and c_k are determined by ensuring that the interpolation passes through all the $n+n_b$ nodes within the support domain:

$$\begin{aligned} R_i(r) &= R(\|\mathbf{x}-\mathbf{x}_i\|) \\ R_j^b(r) &= R(\|\mathbf{x}-\mathbf{x}_j^b\|) \\ \frac{\partial R_j^b(r)}{\partial n} &= l_j^x \frac{\partial R_j^b}{\partial x} + l_j^y \frac{\partial R_j^b}{\partial y} \end{aligned} \tag{2}$$

where l_j^x and l_j^y are the elements of normal vector at the j^{th} point on boundaries.

Polynomial basis has the following monomial terms:

$$\mathbf{P^T(x)} = [1, x, y, x^2, xy, y^2, \cdots] \tag{3}$$

The interpolations of the normal derivatives of function at the j^{th} point on the boundaries have the following form:

$$\frac{\partial u^b(x_j)}{\partial n} \cong \sum_{i=1}^{n} a_i \frac{R_i(r)}{\partial n} + \sum_{j=1}^{n_b} b_j \frac{\partial}{\partial n}\left(\frac{\partial R_j^b(r)}{\partial n}\right) + \sum_{k=1}^{m} c_k \frac{\partial}{\partial n}\big(P_k(x)\big), \; j = 1, 2, \ldots, n_b \tag{4}$$

What's more, the additional polynomial terms have to satisfy an extra requirement of unique approximation. Furthermore, in order to guarantee the system matrix symmetric, the following constraints are generally imposed:

$$\sum_{i=1}^{n} a_i P_k(x_i) + \sum_{j=1}^{n_b} b_j \frac{\partial}{\partial n}\big(P_k(x_j)\big) = 0, k = 1, 2, \ldots, m \tag{5}$$

The matrix form of these equations can be expressed as:

$$\tilde{U}_S = \begin{Bmatrix} U_S \\ \dfrac{\partial U_b}{\partial n} \\ 0 \end{Bmatrix} = \begin{bmatrix} R_0 & R_b & P_m \\ R_b^T & R_c & P_b \\ P_m^T & P_b^T & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = G\tilde{A} \tag{6}$$

where the vector of function values $\tilde{U}_S$ is

$$\tilde{U}_S^T = \begin{Bmatrix} u_1 & \cdots & u_n & \dfrac{\partial u_1^b}{\partial n} & \cdots & \dfrac{\partial u_{n_b}^b}{\partial n} & 0 & \cdots & 0 \end{Bmatrix} \tag{7}$$

Thus, the unknown coefficients vector is found to be

$$\tilde{A} = G^{-1}\tilde{U}_s \tag{8}$$

The form of the approximation function can be obtained as

$$u(x) = \begin{Bmatrix} R^T(x) & \dfrac{\partial R_b^T}{\partial n} & P^T(x) \end{Bmatrix} G^{-1}\tilde{U}_s = \tilde{\Phi}^T(x)\tilde{U}_s \tag{9}$$

The vector of shape function in Eq. (9) can be expressed as follow:

$$\tilde{\Phi}^T(x) = \begin{Bmatrix} R^T(x) & \dfrac{\partial R_b^T}{\partial n} & P^T(x) \end{Bmatrix} G^{-1} = \begin{Bmatrix} \phi_1 & \cdots & \phi_n & \phi_1^H & \cdots & \phi_{n_b}^H & \phi_1^P & \cdots & \phi_m^P \end{Bmatrix} \tag{10}$$

In Eq. (10), ϕ_n can be expressed as

$$\phi_k(r) = \sum_{i=1}^{n} R_i(x)\bar{G}_{i,k} + \sum_{j=1}^{n_b} \frac{\partial R_j^b}{\partial n}\bar{G}_{n+j,k} + \sum_{q=1}^{m} P_q(x)\bar{G}_{n+n_b+q,k}, k = 1,2,\cdots,n \quad (11)$$

where $\bar{G}_{i,k}$ is the (i,k) element of matrix G^{-1}.

Three particular forms of radial basis functions are usually introduced: Multiquadrics Function, Gaussian Function and Thin Plate Spline type. Proposed by Hardy [6] and then developed by other researcher [7,8], Multiquadrics Function (also called MQ) has following expression:

$$R_i(x,y) = (r_i^2 + R^2)^q, R \geq 0 \quad (12)$$

where q and R are shape parameters.

Gaussian Function (also abbreviated as EXP) has the following form:

$$R_i(x,y) = \exp(-\alpha(\frac{r_i}{d_c})^2) \quad (13)$$

where $\alpha(\alpha \geq 0)$ is a shape parameter and d_c is the average distance of neighborhood nodes in the influence domain.

Thin Plate Spline function (also abbreviated as TPS) has following formula:

$$R_i(x,y) = |r_i|^\eta \quad (14)$$

where η is a shape parameter.

With detailed modification for shape parameters and influence domain, it is convenient to develop the Radial basis function to multidimensions. A simulation is performed to reveal the property of RBFs shape function over the support domain $x_i \in [-1,-0.5,0,0.5,1]$ with uniform nodes arrangement.

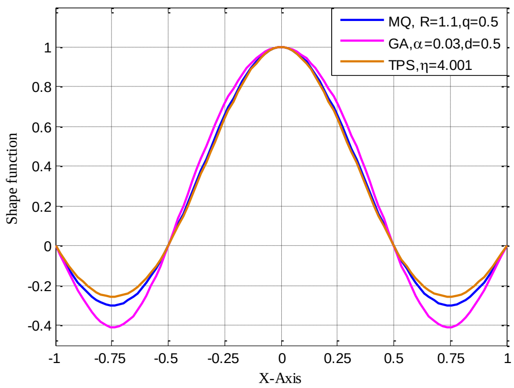


Figure 1: The shape functions of different RBFs at the node $x = 0$

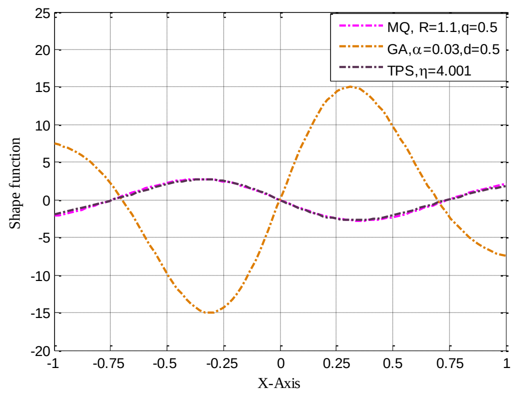


Figure 2: The derivatives of shape functions at the node: $x = 0$

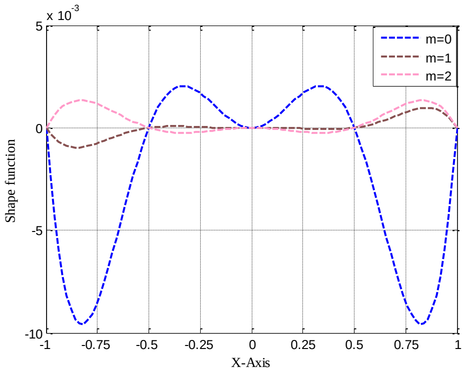


Figure 3: The shape function of polynomial terms at the node: $x = 0$

As observed in Figs. 1-3, these results inherently indicate that the shape functions of Hermite radial point interpolation method with polynomial terms possess linearly consistency and the delta function property, which facilitate convenient implementation of essential boundary conditions.

3. Approximation method

The governing equation and boundary equation of numerous physical can be described as follow

$$\begin{cases} L\big(u(x)\big) = f(x) & x \in \Omega \\ B\big(u(x)\big) = g(x) & x \in \Gamma \end{cases} \tag{15}$$

where Ω and Γ denote the interior computational domain and the boundary domain, respectively. Moreover, L and B represent the differential operators of the governing equation and boundary equations, respectively. What's more, $f(x)$ and $g(x)$ denote the defined conditions of governing equation and boundary equation.

Generally, the boundary conditions can be expressed as

$$B\big(u(x)\big) = \alpha u(x) + \beta \frac{\partial u(x)}{\partial n} \tag{16}$$

where n denotes the outward unit vector normal to the boundary $n = \langle n_x, n_y \rangle^T$ .If $\beta = 0$, Eq.(16) represents Dirichlet boundary condition; if $\alpha = 0, \beta \neq 0$. Eq.(16) yields Neumann boundary condition. Otherwise, namely $\alpha \neq 0, \beta \neq 0$, Eq.(16) is known as Robin boundary condition.

4. Numerical examples

In this section, two examples are provided to show the convergence and stability of RPIM applied in acoustic problems. Example 1 is a case for 1D problem, and then in example 2, the current method is extended to a 2D problem.

Before the detailed description of the examples, the governing equation and criteria are given as follow.

If the fluid in the domain is assumed to be stationary and inviscid, then the problem of acoustic wave propagation in frequency domain can be expressed as Helmholtz equation:

$$(\nabla^2 + k^2)p = 0, \quad \Omega. \tag{17}$$

In the above equation, $k = \omega / c$ is known as the wave number, while ω and c denote the angular frequency and the wave speed, respectively. Besides, ∇^2 is the Laplace operator and Ω is the computational domain.

In case that the exact/analytical solution exists, the convergence rate of the method can be evaluated. The error norm, as an indicator of convergence, is defined as:

$$\varepsilon = \sqrt{\frac{\sum_{i=1}^{N} (p_i^{exact} - p_i^{num})^2}{\sum_{i=1}^{N} (p_i^{exact})^2}} \tag{18}$$

where p_i^{exact} is the analytical solution while p_i^{num} represents the numerical solution at the i^{th} point.

Furthermore, to evaluate the stability of the current approach, the evaluation index can be defined as:

$$\delta = \log 10(cond(G)) \tag{19}$$

where G is the matrix of shape function.

4.1 Example 1

In this example a 1D model is considered to prove the reliability of presented approach and the effect for the number of distributed nodes. In 1D problem, the Eq. (17) and boundary condition can be simplified as:

$$\frac{d^2 p}{dx^2} + k^2 p = 0, x \in [-1,1]$$
$$\left.\frac{\partial p}{\partial x}\right|_{x=-1} = -k\sin(k) \tag{20}$$
$$p|_{x=1} = \cos(k)$$

In this subsection, Dirichlet boundary condition and Neumann boundary condition are adopted as an example. Moreover, the wave number is defined as $k=10$ and the shape parameters are suitably selected according to some numerical experiments.

The results of this example provide comparison of capability and stability for Multiquadrics Function, Gaussian Function and Thin Plate Spline, as shown in Fig.4 and 5.

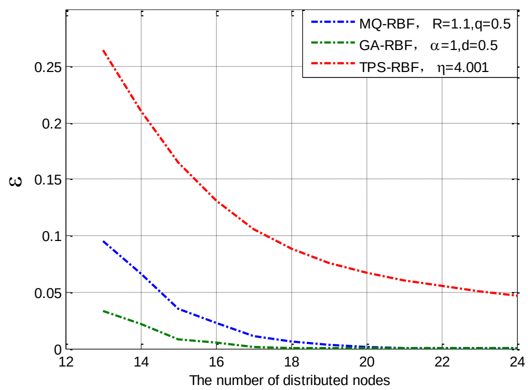


Figure 4: Numerical error of different RBFs along with the number of distributed nodes

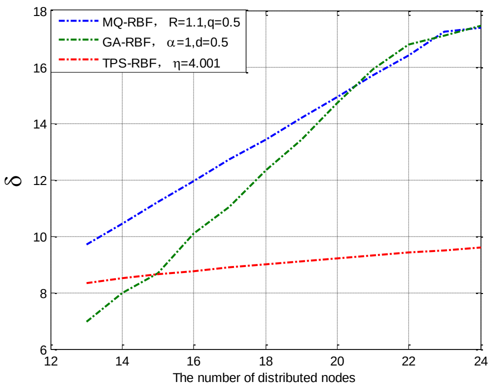


Figure 5: Condition number of shape function matrix along with the number of distributed nodes

These results indicate that the numerical error declines with the increase of the number of distributed nodes. While denser collocation nodes brings larger condition number. Furthermore, from the perspective of errors and stability, Multiquadrics Function and Gaussian Function basis is more suitable to solve Helmholtz equation compared with Thin Plate Spline. However, the stability of Thin Plate Spline function is not so sensitive to the increase of distributed nodes.

What's more, irregular nodes are also adopted in the computational domain to study the adaptive ability of RPIM based on the Hermite collocation method. The comparison shows that the results of the approximated simulation match well with those of the analytical prediction (see Fig. 6).

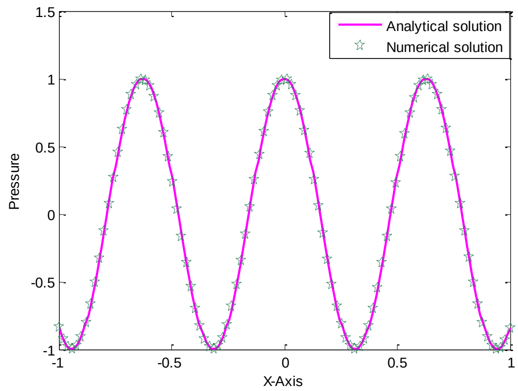


Figure 6: Validation of Hermite radial point interpolation method with 18 irregular nodes

4.2 Example 2

In this example the performance of the method in solving a 2D acoustic problem is investigated. The following set of parameter is used in this subsection：

$$frequency = 500\,\text{Hz}\;,\;\rho = 1.225\,\text{kg/m}^3 \text{ and } c = 340\,\text{m/s}$$

The 2D problem is described as

$$\begin{aligned} &\frac{d^2 p}{dx^2} + \frac{d^2 p}{dy^2} + k^2 p = 0, x \in [0,1], y \in [0,1] \\ &p|_{x=0} = p|_{x=1} = 1 \\ &p|_{y=0} = p|_{y=1} = 1 \end{aligned} \tag{21}$$

For the numerical calculation, a set of regular distributed nodes is used to discretize the computational domain (as shown in Fig.7) and the problem in Eq. (21) is solved by RPIM (as shown in Fig.8). Both RPIM and FEM are used to solve the above problem, and the results are given in Figs. 8 for comparison. As the reference solution, FEM model is built and solved in COMSOL with 12765 triangular elements

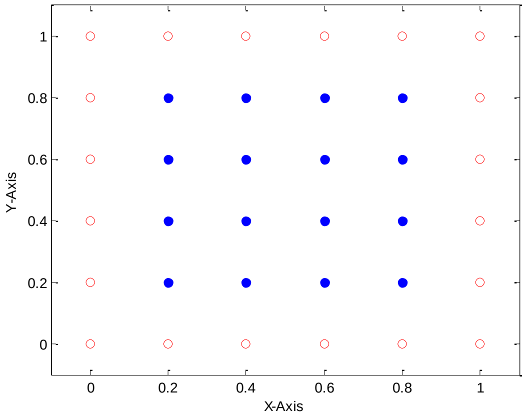
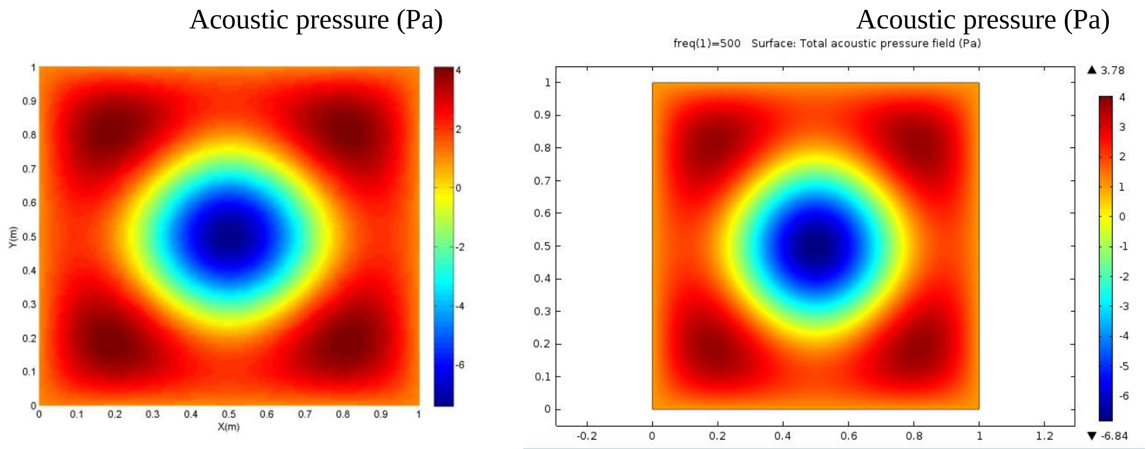
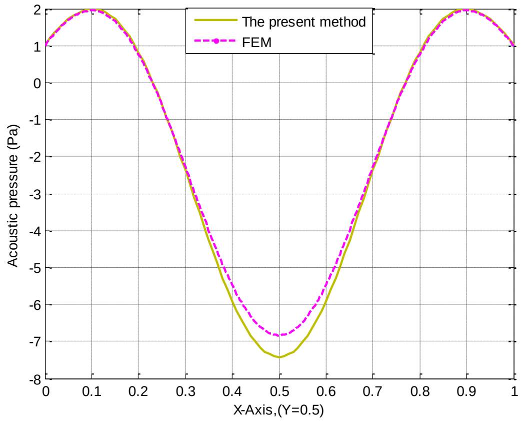


Figure 7: 36 regular nodes in the computational domain



(a) (b)



(c)

Figure 8: Comparison of the acoustic pressure field obtained by (a) the present method, and (b) the FEM in example 2; (c) is extracted at $y = 0.5$ for comparison.

Fig. 8 inherently indicates that the two sets of results are in good agreement, which clearly demonstrates the validation of the present method in solving acoustic problems. And compared with the FEM, RPIM based on the Hermite collocation method can provide acceptable solutions with less distributed nodes. Thus this method has the potential to improve the computational efficiency.

5. Conclusions

Hermite radial point interpolation method with polynomials is applied to give the numerical solution of Helmholtz equation. Two numerical examples are presented to verify the validation of the current approach in solving acoustic problems. In the terms of errors and stability, Gaussian Function basis is more suitable to solve Helmholtz equation compared with Multiquadrics Function and Thin Plate Spline. The fact that irregular node distributions produce rather satisfactory numerical results demonstrates the good adaptive ability of RPIM based on the Hermite collocation method in solving acoustic problems.

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