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Impact Sound From Water Drops

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1. Introduction

In a recent paper [1], we presented a theoretical study of sound generation by the initial impact caused when an idealised spherical water drop vertically falls upon an otherwise quiescent water surface. That study makes use of the simple symmetrical geometry in the initial impact process so that Kirchhoff theorem can be applied and the generated waves and the energy carried by them can be evaluated in closed forms. This enabled us to show that the rapid momentum exchange between the fluid in the falling drop and that in the main water body causes the radiation of compressive waves. These waves are radiated in the form of a wave packet with a densely packed edge which is heard in the far field as a noisy shock-like pulse followed by a quickly decreasing tail. The wave packet carries with it sound energy proportional to the kinetic energy of the falling drop and to the cube of the impact Mach number. Possible applications of these analytic results to the study of noise from natural rains were also discussed there, and an illustrative example was given in which the noise level due to rain showers is linearly related to the rainfall rate, which was shown to be consistent with observations.

From recent studies on rain noise [2-5], it is clear that a falling droplet causes acoustic radiation essentially through two mechanisms, namely, the initial impact and the subsequent entrainment and pulsations of air bubbles. For the impact sound, it has been observed that many factors may affect to the radiated sound. These includes, for example, the statistic distribution of the drop size, the shape of the drops, the impact velocities and the angle at which a drop enters the water body [5-7]. Though our previous study [1] gives an illustration where impact noise is studied analytically, it remains to be seen whether that study can be extended to cases containing the above-mentioned factor, which is essential to being able to make detailed comparison between our theoretical results and observations. The work reported here is a small step towards this direction, involving shape changes from a spherical drop. While we will present the theory for arbitrary drop shapes, particular emphasis will be put on ellipsoid shaped drops which have been observed to closely approximate drops in natural rains [8].

The formulation for the pressure fluctuations generated by the impact of an arbitrarily shaped drop is given in section 2, which is done by using the Kirchhoff theorem to express the pressure in terms of the normal velocity distribution on the surface of the main water body. With this formulation, it is shown that the solutions for an elliptical drop can be infered from those of a spherical drop by simple algebraic manipulations. The ellipsoid shaped drop is simply equivalent to a spherical one at a higher impact velocity and with a radius equal to the length of the half major axis of the ellipsoid (the axis parallel to the surface of the water body). The equivalent impact velocity is higher than the actual one by the ratio of the lengths of the two axes. From this relation, the energy radiated by an elliptical drop can be derived. It is shown that an ellipsoid is more efficient in radiating sound than a spherical drop. Suppose that an elliptical drop has the same impact velocity and has the same volume as a spherical drop so that the kinetic energies carried by the two are equal. Because of the shape difference, however, more energy carried by the elliptical drop is radiated to the far field as sound than that of the spherical drop. The difference in the acoustic radiation between the two cases is a factor equal to the sixth power of the ratio between the two axes of the ellipsoid.

2. Formulation

Consider a drop of axisymmetrical shape vertically falling onto an initially quiescent water surface. We choose the initially undisturbed water surface as the coordinate plane $x_3 = 0$, where the Cartesian coordinate system (x_1,x_2,x_3) is fixed such that the drop is falling in the negative x_3 direction with constant velocity U and it touches the water surface at the origin of the coordinate system at $t = 0$. The geometry is illustrated in figure 1. According to the Kirchhoff theorem, the pressure fluctuations at the observation position $\mathbf{x}$ and time t can be expressed in terms of a retarded integral of the normal velocity distribution over a control surface [9]. Choosing the plane $x_3 = 0$ as the contral surface and denoting by $u_3(x_\alpha, t)$ the velocity distribution in the negative x_3 direction on this surface, the Kirchhoff theorem then gives the pressure fluctuations $p(\mathbf{x}, t)$ as

$$p(\mathbf{x},t) = \frac{\rho_0}{2\pi}\frac{\partial}{\partial t}\int_{y_\alpha} \frac{u_3(y_\alpha,\tau)}{|\mathbf{x}-\mathbf{y}|} d^2 y_\alpha, \tag{2.1}$$

where ρ_0 is the constant mean density in water and the integrals are performed on the plane $y_3 = 0$ at the retarded time $\tau = t - |\mathbf{x}-\mathbf{y}|/c$, c being the constant sound speed in water and $|\mathbf{x}-\mathbf{y}|$ the modulus of $\mathbf{x}-\mathbf{y}$. Since the only time dependence in the integrand is contained in τ, the derivative with respect to t can be transfered inside the integrals and converted to one with respect to τ. We can then rewrite (2.1) as

$$p(\mathbf{x},t) = \frac{\rho_0}{2\pi}\int_{y_\alpha} \frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial u_3(y_\alpha,\tau)}{\partial \tau} d^2 y_\alpha. \tag{2.2}$$

To evaluate this, it is necessary to specify the velocity distribution $u_3(x_\alpha, t)$. It can be noted that, during the initial impact, this velocity distribution is nonzero only within the contact circle between the drop and the water surfaces because the contact circle expands supersonically. During this expansion, no waves can travel ahead of the contact

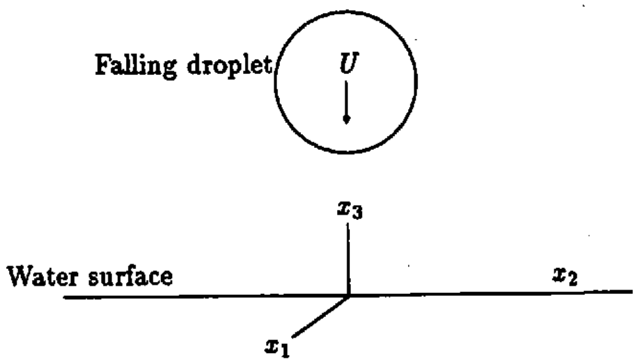

Figure 1. A water droplet falling onto an originally quiescent water surface.

circle so that the waves form a wave packet within the region inside it, as schemetically shown in figure 2. These waves are not aware of any boundary effects so that they are completely anti-sysmetrical about the plane $x_3 = 0$; the velocity within the wave packet varies continuously from zero at the edge of the packet in the water body to the constant value U at the edge in the drop. Thus, the velocity distribution on the median plane must be half the velocity difference U. This is true as long as the expansion velocity of the contact circle is larger than the speed of sound in water; when the expansion velocity becomes subsonic, some waves may overtake the expanding contact circle, be reflected by the pressure release surface ahead of the contact circle, and hence, destroy the complete anti-sysmetry. However, the process taking place at this later stage can be expected to be much less significant in radiating sound waves. On this account, the velocity distribution $u_3(x_\alpha, t)$ can be written as

$$u_3(x_\alpha, t) = \frac{U}{2} H(t) H(b(t) - |x_\alpha|), \tag{2.3}$$

where $b(t)$ is the radius of the drop/surface contact circle and H is the Heaviside step function, equal to unity for positive arguments and zero otherwise. The time dependent contact curve between the drop and the water body has been denoted by the simple circle because of the assumption that the drop is axisymmetrical; if the drop is of other shape or if it is falling obliquely onto the water surface, this axisymmetry would be no longer assumed so that a more general expression would have to be used to replace the argument $b(t) - |x_\alpha|$ in the second Heaviside function in (2.3).

On substituting (2.3) into the Kirchhoff formulation (2.2), it immediately follows that

$$p(\mathbf{x}, t) = \frac{\rho_0 U}{4\pi} \int_{y_\alpha} \frac{1}{|\mathbf{x} - \mathbf{y}|} \frac{\partial b(\tau)}{\partial \tau} H(\tau) \delta(b(\tau) - |y_\alpha|) d^2 y_\alpha, \tag{2.4}$$

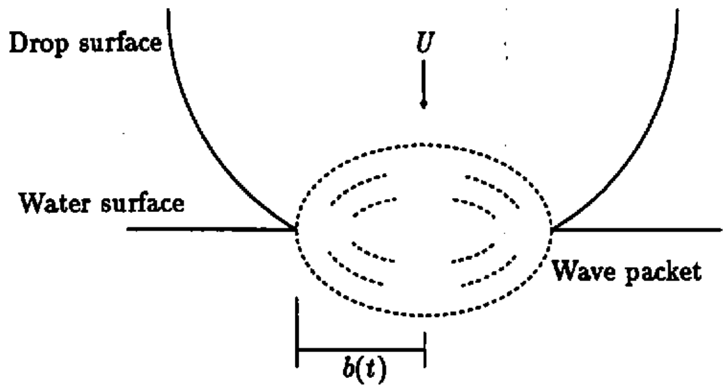

Figure 2. The wave packet produced by the supersonically expanding contact circle during the initial impact.

where δ is the Dirac delta function resulting from the derivative of the Heaviside function. The y_α integrals can be conveniently evaluated by changing the integration variable to polar coordinates according to

$$y_1 = \lambda \cos\alpha \quad \text{and} \quad y_2 = \lambda \sin\alpha, \tag{2.5}$$

and performing the α integral by making use of the properties of the δ function. When this is done, the pressure fluctuations can be found to be

$$p(\mathbf{x},t) = \frac{\rho_0 U}{4\pi}\int_0^\infty \frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial b(\tau)}{\partial \tau}\left|\frac{\partial b(\tau)}{\partial \alpha}\right|^{-1} H(\tau)\lambda d\lambda. \tag{2.6}$$

This result can be further simplified by noticing the chain rule of differentiation

$$\left|\frac{\partial b(\tau)}{\partial \alpha}\right| = \left|\frac{\partial b(\tau)}{\partial \tau}\frac{\partial \tau}{\partial \alpha}\right| = \frac{\partial b(\tau)}{\partial \tau}\frac{\lambda|x_\alpha|}{c|\mathbf{x}-\mathbf{y}|}|\sin(\alpha-\alpha')|, \tag{2.7}$$

where $\alpha' = \arctan(x_2/x_1)$ and the last step follows from using the retarded time $\tau = t - |\mathbf{x}-\mathbf{y}|/c$ to calculate $\partial\tau/\partial\alpha$. When this result is substituted into (2.6), we derive

$$p(\mathbf{x},t) = \frac{\rho_0 U c}{4\pi|x_\alpha|}\int_0^\infty \frac{H(\tau)}{|\sin(\alpha-\alpha')|}d\lambda, \tag{2.8}$$

where α is now a function of the integration variable λ, determined by the implicit equation $b(\tau) - \lambda = 0$ with b given by the shape of the drop.

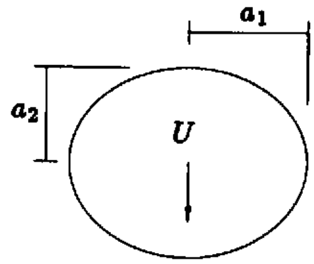

Figure 3. An ellipsoid water drop whose vertical falling makes $a_1 > a_2$.

3. The transform from spherical to elliptical shape

The radius of the contact circle $b(t)$ can be found from the geometry of the problem once the shape of the drop is given. For example, in our previous study [1] where the drop was assumed to be a sphere of radius a, $b(t)$ is then given by

$$b(t) = \left[a^2 - (a - Ut)^2\right]^{1/2}, \tag{3.1}$$

from which, the expanding velocity of the circle can be derived by differentiation as

$$\dot{b}(t) = \frac{(a - Ut)U}{[a^2 - (a - Ut)^2]^{1/2}}, \tag{3.2}$$

where the overhead dot denotes a time derivative. This is larger than the constant sound speed c for small time. The instant t_c at which $\dot{b}$ becomes smaller than c is given by

$$t_c = \frac{a}{U}\left(1 - \frac{1}{\sqrt{1 + M^2}}\right) \sim \frac{1}{2}M^2\frac{a}{U} \quad \text{for} \quad M \ll 1, \tag{3.3}$$

where $M = U/c$ is the small impact Mach number.

In natural rains, the shape of a falling drop depends on many factors. It has been observed that ellipsoid is a very close approximation [8]. Because of its falling motion, a drop has higher surface pressure in the forward and backward regions on its surface, and lower pressure on the sideward regions. As a result, the drop is flattened so that its axis parallel to the direction of motion becomes sharter than those in the other two directions. This gives an elliptical shape illustrated in figure 3. Denoting the half lengths of the two axes respectively by a_1 and a_2 with $a_1 > a_2$, the geometry of the problem leads to

$$b(t) = \frac{a_1}{a_2}\left[a_2^2 - (a_2 - Ut)^2\right]^{1/2}. \tag{3.4}$$

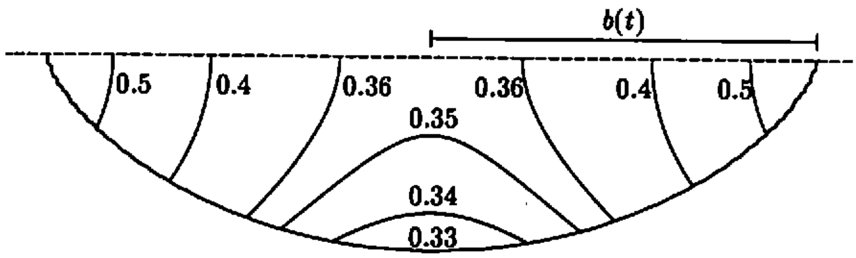

Figure 4. Contours of $p(\mathbf{x},t)/\rho_o Uc$ within the wave packet.

Its time derivative then gives the expansion velocity

$$\dot{b}(t) = \frac{a_1}{a_2}\frac{(a_2 - Ut)U}{[a_2^2 - (a_2 - Ut)^2]^{1/2}}, \tag{3.5}$$

and the critical time for $\dot{b}(t)$ equal to c can be found to be

$$t_c = \frac{a_2}{U}\left(1 - \frac{1}{\sqrt{1 + M^2 a_1^2/a_2^2}}\right) \sim \frac{1}{2}M^2\frac{a_2}{U}\left(\frac{a_1}{a_2}\right)^2 \quad \text{for} \quad M \ll 1. \tag{3.6}$$

Comparing the set of results for a spherical drop with those for an ellipsoid, it is clear that (3.4), (3.5) and (3.6) can be expressed by the results (3.1), (3.2) and (3.3) provided that a and U in the latter case are replaced respectively by a_1 and $U' = Ua_1/a_2$. Note also that the sound pressure (2.8) is completely determined once $b(t)$ is given. Thus, the sound from an elliptical drop can be easily infered from the solution for a spherical drop. The above results actually give an analogy between the two cases; an ellipsoid drop is equivalent to a spherical one with an analogeous radius a_1 and a higher impact velocity U' (because $a_1 > a_2$).

4. The pressure waves

For a spherical drop, we have calculated the pressure (2.8) with the result [1]

$$p(\mathbf{x},t) = \frac{\rho_0 Uc}{\pi}\int_0^{b(t)} \frac{H\left[4\lambda^2 x_\alpha^2 - (|\mathbf{x}|^2 + \lambda^2 - h^2)^2\right]}{\left[4\lambda^2 x_\alpha^2 - -(|\mathbf{x}|^2 + \lambda^2 - h^2)^2\right]^{1/2}}\lambda d\lambda, \tag{4.1}$$

where h is introduced to save writing; $h = (Ut - a + \sqrt{a^2 - \lambda^2})/M$. This can be rewritten in a more explicit form, namely,

$$\frac{p(\mathbf{x},t)}{\rho_0 Uc} = \frac{1}{\pi}\int_0^{\tilde{t}} \frac{(1-M^2\eta)}{\left\{4\tilde{x}_\alpha^2\eta(2-M^2\eta) - \left[\tilde{\mathbf{x}}^2 + \eta(2-M^2\eta) - (\tilde{t}-\eta)^2\right]^2\right\}^{1/2}} d\eta, \tag{4.2}$$

where the overhead tildes means nondimensional quantities with the definitions

$$\tilde{t} = \frac{Ut}{aM^2} \quad \text{and} \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{aM}, \tag{4.3}$$

and the integration limits are also determined by requiring the argument in the square root to be positive, which comes from the Heaviside function in (4.1). The argument in the square root is a quartic polynomial in the integration variable η. Thus, according to the formulae (3.148) of [10], the integral is in principle expressible in terms of elliptic functions. It is also straightforward to evaluate it numerically; some results are shown in figure 4 where contours of $p(\mathbf{x},t)/\rho_0 Uc$ are plotted in the $(|x_\alpha|, x_3)$ plane, corresponding to the case of $\tilde{t} = 0.5$ and $M = 10^{-3}$. Since the waves are anti-sysmetrical about the plane $x_3 = 0$, only half of the wave packet is plotted. The details of this result has been analysed in our previous study [1]. Here we only point out that though this is calculated for a spherical drop, the features analysed in [1] all pertain to the case of an elliptical drop. For example, the mechanism by which compressive waves are radiated is the same for the two cases and the form of the generated wave packet is also the same. This is guaranteed by the analogy between the two cases shown in the previous section.

5. The radiated acoustic energy

As analysed in [1], the waves shown in figure 4 must undergo reflections from the boundary surfaces before they escape to the far field, so that the far field radiation pattern can only be examined by including these reflection effects, which does not appear to be an easy task. However, the acoustic energy radiated to the far field can be calculated directly from the results derived in the previous sections without having to examine the far field structure. This is because the energy radiated to infinity is precisely equal to the energy carried by the waves shown in figure 4 and represented by the result (2.8); the reflections only affect the form of pressure pulse. There is no mechanism in our model for this compressional energy, once it has been converted from the kinetic energy of the falling drop, to be converted back to kinetic energy again, so that it must all be radiated to the far field. Thus, the radiated acoustic energy can be calculated from the formula

$$E = 2\int_0^{t_0}\int_{x_\alpha} p(x_\alpha, 0, t) u_3(x_\alpha, t) dt d^2x_\alpha, \tag{5.1}$$

where $u_3(x_\alpha, t)$ is the velocity distribution on the plane $x_3 = 0$ specified by (2.3) and $p(x_\alpha, 0, t)$ can be conveniently taken as (2.8) or (4.1) with x_3 set to zero. The factor 2 in (5.1) takes account of the energy carried by those waves which originally propagate upwards into the droplet, but are later reflected back into the water body by the drop surface and eventually radiated to the far field.

Following [1] by substituting (2.3) and (4.1) into (5.1), we derive

$$E = 2\rho_0 U^2 c \int_0^{t_c} \int_0^{b(t)} \int_0^{b(t)} \frac{H\left[4\lambda^2 r^2 - (r^2+\lambda^2-h^2)^2\right]}{\left[4\lambda^2 r^2 - (r^2+\lambda^2-h^2)^2\right]^{1/2}} \lambda d\lambda r dr dt, \tag{5.2}$$

which follows from changing the x_α integrals to polar coordinates according to $x_1 = r\cos\beta$ and $x_2 = r\sin\beta$ with the β integral equal trivially to 2π since the integrand has no β dependence. By making use of the formula (2.261) of [10], the r integral can be carried out with the result

$$\int_0^{b(t)} \frac{H\left[4\lambda^2 r^2 - (r^2+\lambda^2-h^2)^2\right]}{\left[4\lambda^2 r^2 - (r^2+\lambda^2-h^2)^2\right]^{1/2}} r dr = \frac{1}{2}\arcsin \frac{r^2-\lambda^2-h^2}{2\lambda|h|}\bigg|_A^B, \tag{5.3}$$

where the integration bounds A and B have been shown in [1] to be

$$A = |\lambda - |h|| \quad \text{and} \quad B = \lambda + |h|. \tag{5.4}$$

On substituting these into (5.3), the right hand side of it reduces to $\pi/2$ and (5.2) becomes

$$E = \rho_0 U^2 c\pi \int_0^{t_c} \int_0^{b(t)} \lambda d\lambda dt = \frac{\pi}{2}\rho_0 U^3 c t_c^2 \left(a - \frac{1}{3}U t_c\right). \tag{5.5}$$

Using (3.3) for t_c, this can be rewritten in terms of the impact Mach number M as

$$\frac{E}{E_0} = \frac{1}{4M}\left(2 + \frac{1}{\sqrt{1+M^2}}\right)\left(1 - \frac{1}{\sqrt{1+M^2}}\right)^2 \sim \frac{3}{16}M^3 \quad \text{for} \quad M \ll 1, \tag{5.6}$$

where E_0 is the kinetic energy carried by the falling drop, that is, $E_0 = (4/6)\rho_0 U^2 \pi a^3$.

The result (5.6) is calculated for a spherical drop. As for the pressure fluctuations, if an elliptical drop is considered, the radiated acoustic energy for such an ellipsoid can be infered from (5.6) be simply replacing a by a_1 and U by Ua_1/a_2. This leads to the result

$$E_e = \frac{\pi\rho_0}{8c^3} U^5 a_1^3 \left(\frac{a_1}{a_2}\right)^5, \tag{5.7}$$

where we have used the subscript e to indicate the case of an elliptical drop. It is of interest now to examine the radiation efficiency of elliptical drops in comparison to spherical drops.

To this end, we assume that the elliptical drop has the same impact velocity and the same volume as a spherical drop of radius a so that the kinetic energies carried by the two drops are the same. This requires the equal volume relation $a^3 = a_1^2 a_2$. From the results (5.6) and (5.7), it immediately follows that

$$\frac{E_e}{E} = \left(\frac{a_1}{a_2}\right)^5 \frac{a_1^3}{a^3} = \left(\frac{a_1}{a_2}\right)^6, \qquad (5.8)$$

where the last step follows from the use of $a^3 = a_1^2 a_2$. Since $a_1 > a_2$, it is then clear that more energy is radiated as sound in the case of an elliptical drop than a spherical drop.

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